# Overview



Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

Linear Temporal Logic (LTL)

Computation-Tree Logic

**Equivalences and Abstraction**

- bisimulation
- CTL, CTL*-equivalence
- computing the bisimulation quotient ⟵
- abstraction stutter steps
- simulation relations

# Bisimulation quotient $\mathcal{T}/\!\sim$



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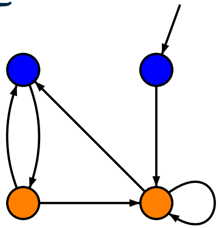

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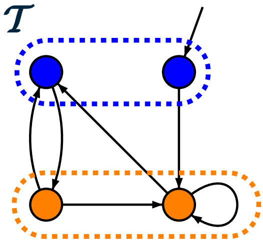

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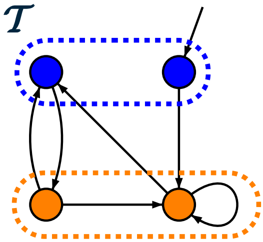


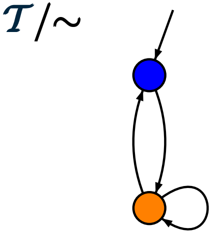

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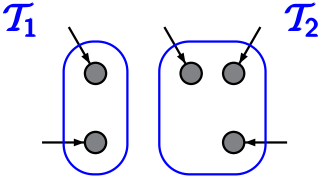

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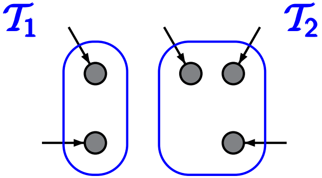


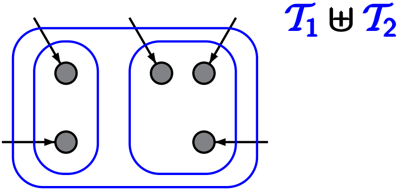

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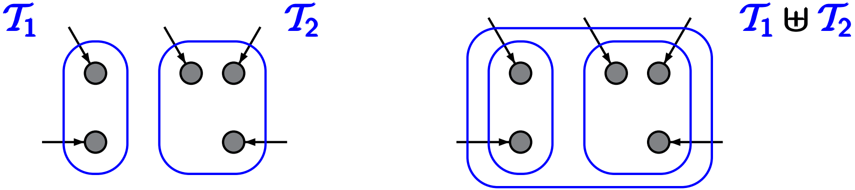

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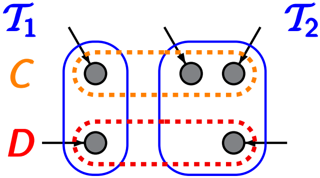


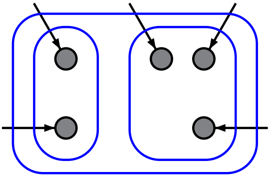

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   replace $\mathcal{T}$ with $\mathcal{T}/{\sim}$ and analyze $\mathcal{T}/{\sim}$

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*here*: only explanations for finite transition systems, possibly with terminal states

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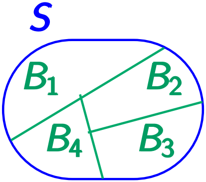


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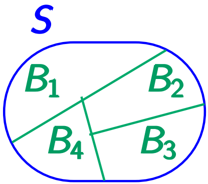


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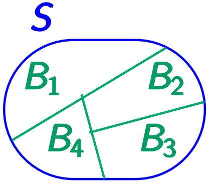


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A superblock denotes any union of blocks.

# Partitions and equivalences



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- equivalence $\mathcal{R}$ on $S \rightsquigarrow$ partition $\mathcal{B} = S/\mathcal{R}$

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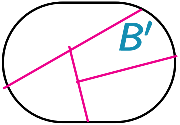
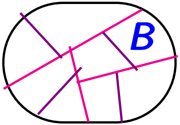



$\mathcal{B}_2$

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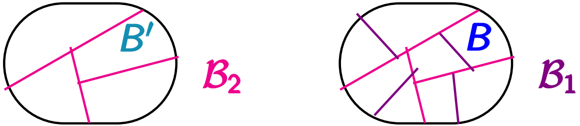


*Example:* if $\mathcal{R}$ is a bisimulation for $\mathcal{T}$ and an equivalence then $S/\mathcal{R}$ is *finer* than $S/\sim$

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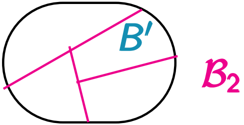


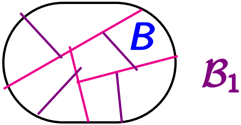


$\mathcal{B}_1$ is called *strictly finer* than $\mathcal{B}_2$ if

(1) $\mathcal{B}_1$ is finer than $\mathcal{B}_2$ and (2) $\mathcal{B}_1 \neq \mathcal{B}_2$

# Computing the bisimulation quotient



by stepwise refinement of partitions the state set $S$

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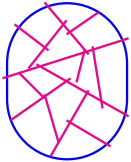

$S/\!\sim$

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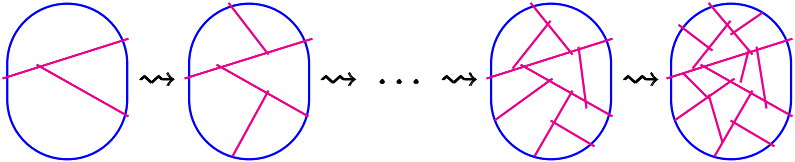

$\mathcal{B}_0 \rightsquigarrow \mathcal{B}_1 \rightsquigarrow \ldots \rightsquigarrow \mathcal{B}_{k-1} \rightsquigarrow \mathcal{B}_k = S/\sim$

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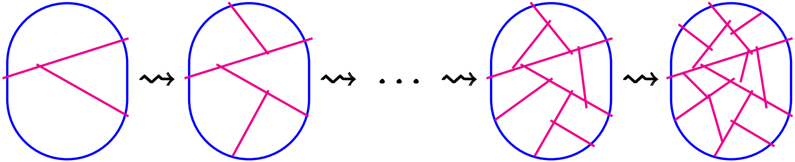

$\mathcal{B}_{AP} = \mathcal{B}_0$ $\quad \mathcal{B}_1$ $\quad \mathcal{B}_{k-1}$ $\quad \mathcal{B}_k = S/\!\sim$

initial partition: $\mathcal{B}_{AP} = \mathcal{B}_0$

identifies all states with the same labeling

# Computing the bisimulation quotient

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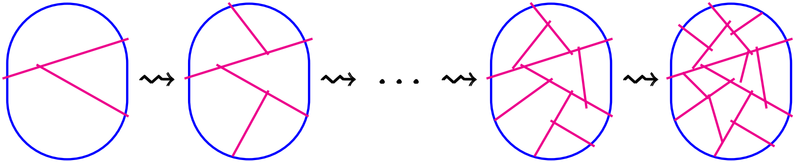

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initial partition: $\mathcal{B}_{AP} = \mathcal{B}_0 = S/\mathcal{R}_{AP}$ where

$$\mathcal{R}_{AP} = \{ (s_1, s_2) : L(s_1) = L(s_2) \}$$

# Characterization of $S/{\sim_{\mathcal{T}}}$



... as the coarsest partition of the
state space $S$ such that ....

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where $Pre(C) = \{ s \in S : \exists s' \in C \text{ s.t. } s \to s' \}$

# Partitioning refinement algorithm



*input:* finite TS $\mathcal{T}$ with state space $S$ over $AP$ (possibly with terminal states)

*output:* bisimulation quotient $S/\sim_{\mathcal{T}}$

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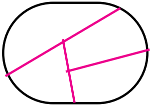

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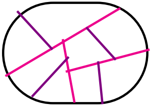

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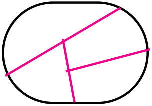
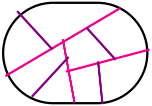
$\mathcal{B}_i$ 

$\mathcal{B}_{i+1}$ 

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hence: $\mathcal{B}_i = S/{\sim_\mathcal{T}}$

$\mathcal{B}_i$ $\mathcal{B}_{i+1}$

# Partitioning refinement algorithm



$\mathcal{B}_0 := \mathcal{B}_{AP}$ ← identifies states with the same labeling

$i := 0$

REPEAT $\mathcal{B}_{i+1} := Refine(\mathcal{B}_i)$

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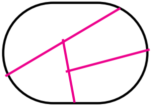
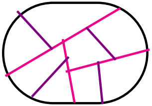
$\mathcal{B}_i$ 

$\mathcal{B}_{i+1}$ 

loop invariant:

$\mathcal{B}_i$ is coarser than $S/\sim_{\mathcal{T}}$ and finer than $\mathcal{B}_{AP}$

# Maximal number of iterations?



$\mathcal{B}_0 := \mathcal{B}_{AP}$; $i := 0$

```
REPEAT
```

$\mathcal{B}_{i+1} := \mathit{Refine}(\mathcal{B}_i)$; $i := i+1$

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Assuming that $\mathcal{B}_i$ is strictly coarser than $\mathcal{B}_{i+1}$ for all $i$, what is the maximal number of refinement steps **?**

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**answer:** $|S| - 1$

Note that $|\mathcal{B}_i| \geq i+1$.

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**answer:** $|S| - 1$

Note that $|\mathcal{B}_i| \geq i+1$.

Hence: if there are $k = |S|-1$ iterations then $\mathcal{B}_k$ consists of singletons

# The initial partition



initial partition $\mathcal{B}_{AP}$:

identifies all states $s$, $t$
s.t. $L(s) = L(t)$

# The initial partition



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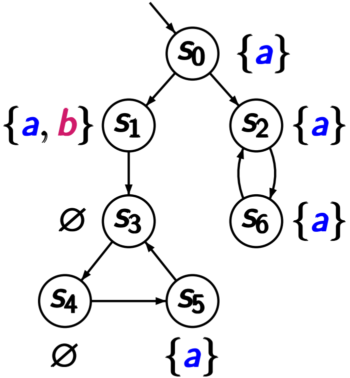

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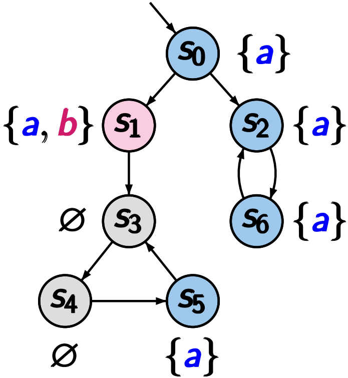


$$\mathcal{B}_{AP} = \Big\{ \{s_0, s_2, s_6, s_5\}, \{s_1\}, \{s_3, s_4\} \Big\}$$

# Initial partition



initial partition $\mathcal{B}_{AP}$:

- identifies all states with the same labeling
- agrees with the quotient under the equivalence

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compute $\mathcal{B}_{AP}$ by an on-the-fly generation of the decision tree for $AP$

# Initial partition



compute $\mathcal{B}_{AP}$ by an on-the-fly generation of the decision tree for $AP = \{a_1, ..., a_k\}$

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compute $\mathcal{B}_{AP}$ by an on-the-fly generation of the decision tree for $AP = \{a_1, ..., a_k\}$

*inner nodes* at level $i$: decision "$a_i \in L(s)$ ?"

*leaves:* sets of states with the same labeling compute $\mathcal{B}_{AP}$ by an on-the-fly generation of the decision tree for $AP = \{a_1, \ldots, a_k\}$

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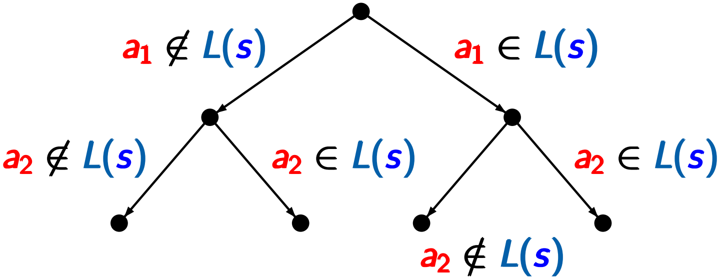


## Computing the initial partition



compute $\mathcal{B}_{AP}$ by an on-the-fly generation of the decision tree for $AP = \{a_1, \ldots, a_k\}$

# Computing the initial partition



compute $\mathcal{B}_{AP}$ by an on-the-fly generation of the decision tree for $AP = \{a_1, \dots, a_k\}$

initally: each leaf represents the empty state-set

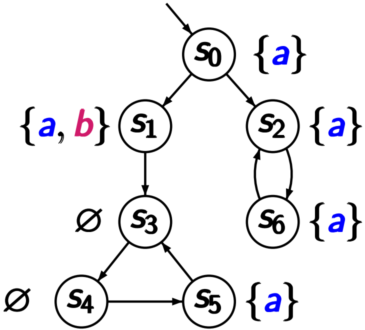
for each state $s$:

traverse the decision tree from the root to a leaf $v$

insert $s$ in the set for $v$

# Example: initial partition





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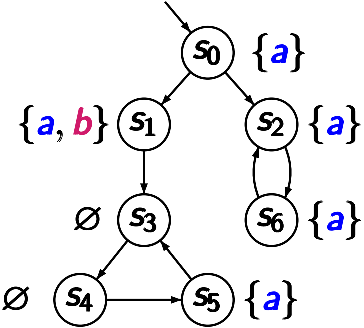


decision tree for
$AP = \{a, b\}$

1. level: $a \in L(s)$ ?
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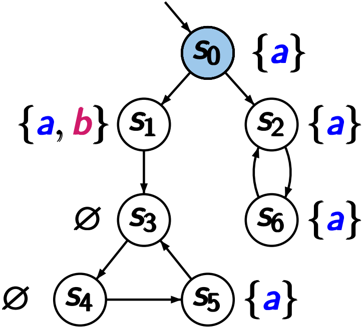


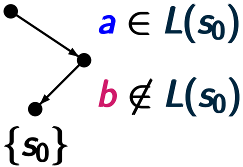


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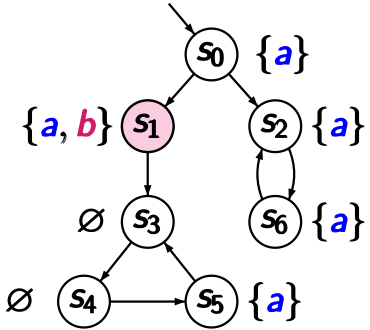


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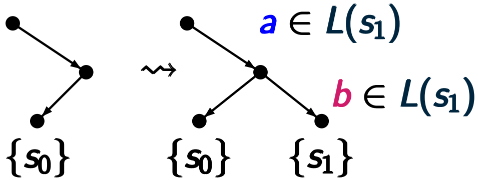

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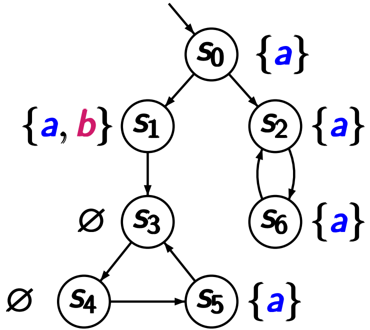


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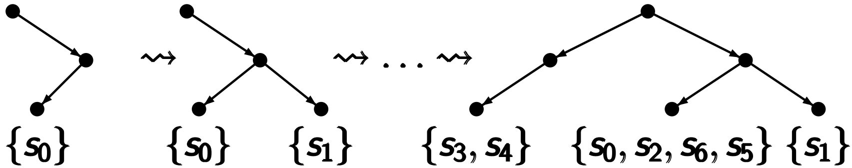

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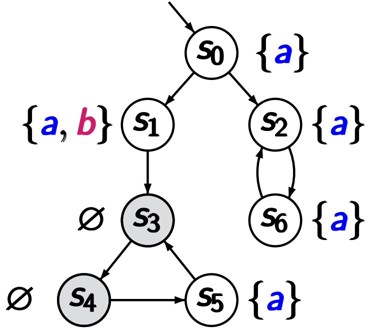


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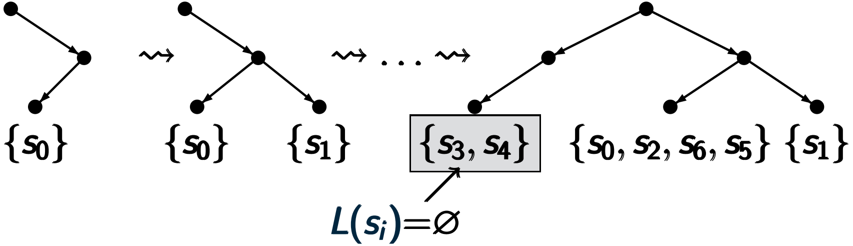

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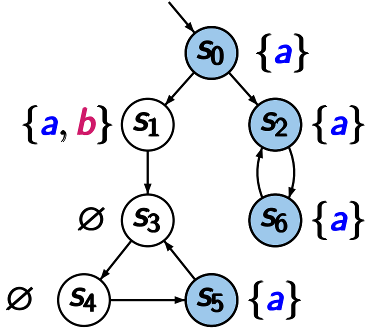


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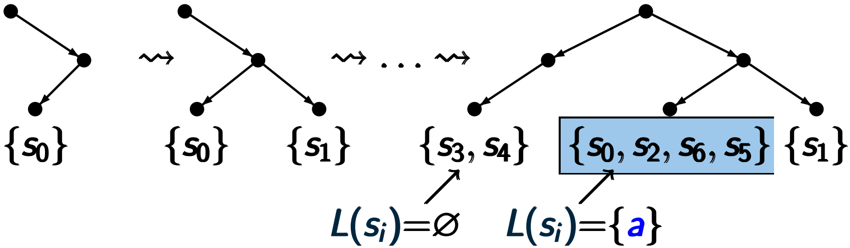

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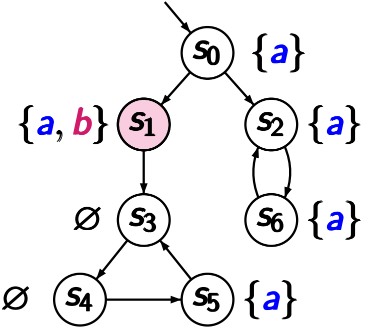


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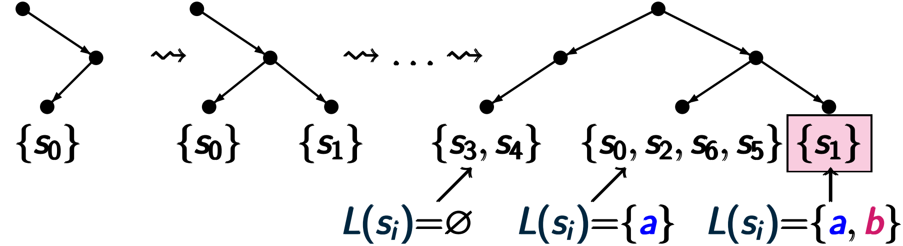

# Computation of the initial partition

```
generate the root node v_0 of the decision tree
FOR ALL states s DO
   v := v_0
   FOR i = 1, ..., k OD
      IF a_i ∈ L(s)
         THEN   v := find_or_add(right son of v)
         ELSE   v := find_or_add(left son of v)
      FI
   OD

OD
```

suppose
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The state-sets of the leaves are the blocks in $\mathcal{B}_{AP}$.

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**complexity:**
$\mathcal{O}(|S| \cdot |AP|)$

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# Partitioning refinement (schema)



$\mathcal{B} := \mathcal{B}_{AP}$

`WHILE` refinements are possible `DO`

$\qquad \mathcal{B} := \mathit{Refine}(\mathcal{B})$

`OD`

return $\mathcal{B}$

# Partitioning refinement (schema)



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```
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return $\mathcal{B}$ ⟵ $\boxed{\mathcal{B} = S/{\sim_{\mathcal{T}}}}$

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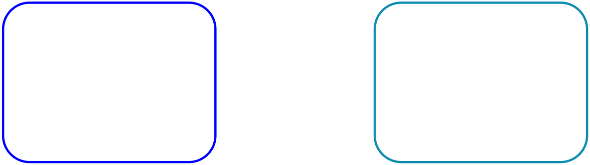
$Refine(B, C)$



block $B$ superblock $C$

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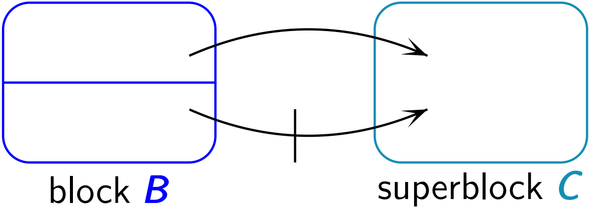

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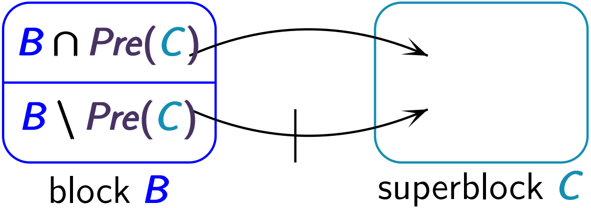
$$Refine(B, C) = \{B \cap Pre(C), B \setminus Pre(C)\}$$



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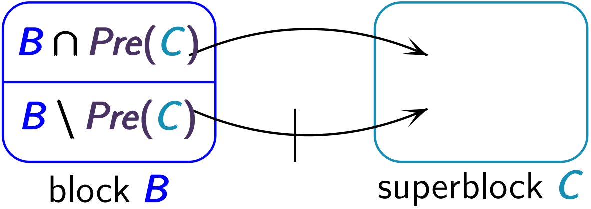

## The refinement operator



Let $\mathcal{B}$ be a partition for $S$ and $C$ a superblock of $\mathcal{B}$.

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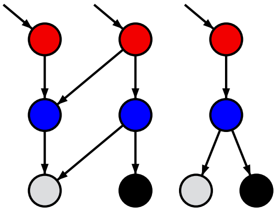
(c) $Refine(\mathcal{B}, C) = \mathcal{B}$ for all $C \in \mathcal{B}$ iff $\mathcal{B} = S/{\sim_{\mathcal{T}}}$

# Example: partitioning splitter algorithm





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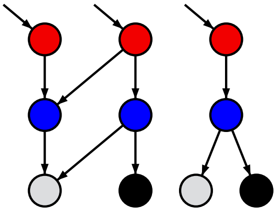

$\Longrightarrow$

refinement
w.r.t. ●

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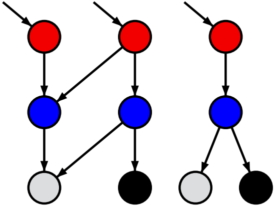

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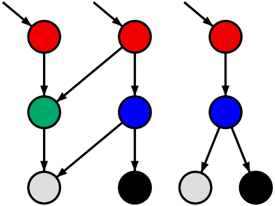

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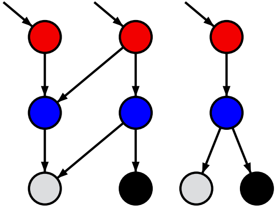

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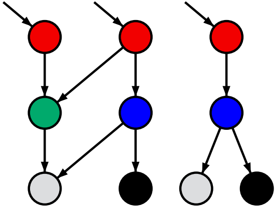
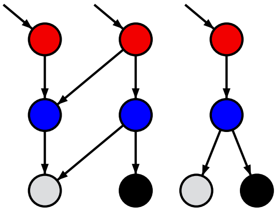


$\Downarrow$

refinement
w.r.t.

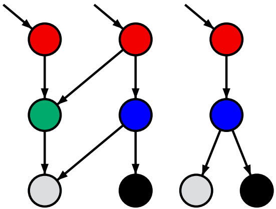
# Example: partitioning splitter algorithm





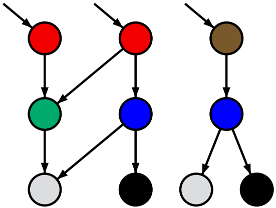
$\Longrightarrow$

refinement
w.r.t.



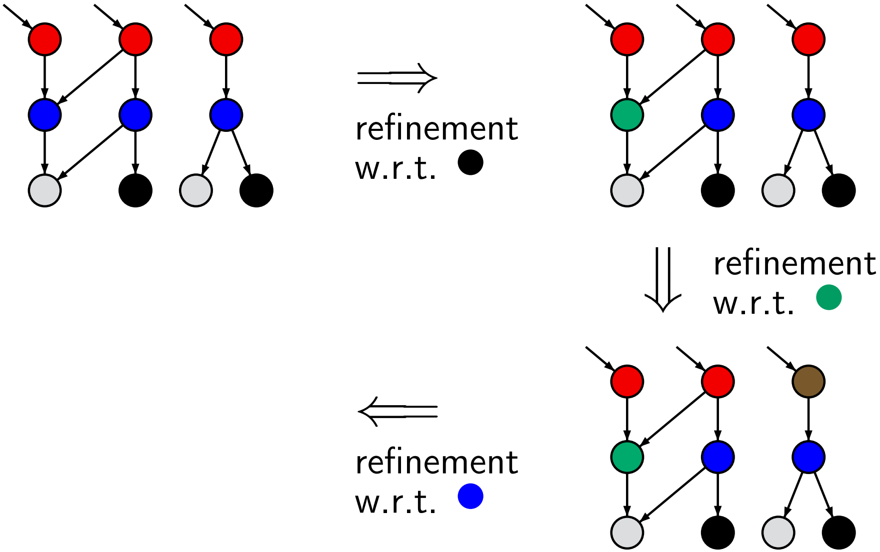
$\Downarrow$

refinement
w.r.t.



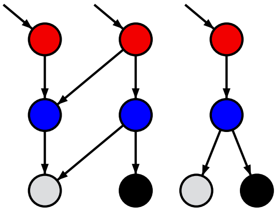
# Example: partitioning splitter algorithm





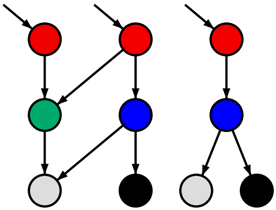
# Example: partitioning splitter algorithm





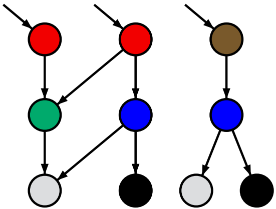
$\Longrightarrow$

refinement
w.r.t.



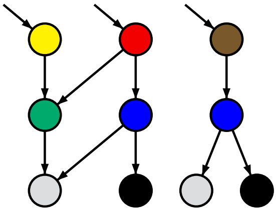
$\Downarrow$

refinement
w.r.t.



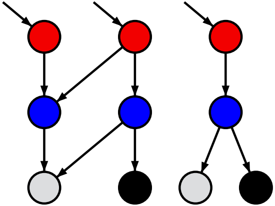
$\Longleftarrow$

refinement
w.r.t.



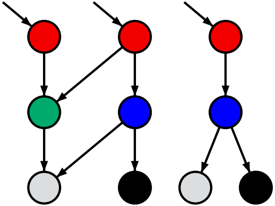
# Example: partitioning splitter algorithm





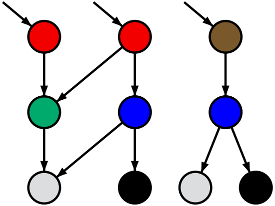
$\Longrightarrow$

refinement
w.r.t.



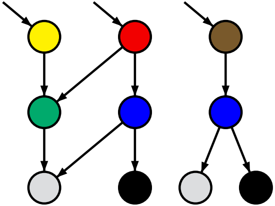
$\Downarrow$

refinement
w.r.t.



$\Longleftarrow$

refinement
w.r.t.



**7** bisimulation equivalence classes

# The refinement operator



given a partition $\mathcal{B}$ and a superblock $C$ of $\mathcal{B}$,
how to compute

$$\mathit{Refine}(\mathcal{B}, C)$$

efficiently **?**

# The refinement operator



given a partition $\mathcal{B}$ and a superblock $C$ of $\mathcal{B}$,
how to compute

$$Refine(\mathcal{B}, C) = \bigcup_{B \in \mathcal{B}} Refine(B, C)$$

efficiently **?**

# The refinement operator

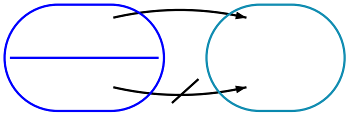
given a partition $\mathcal{B}$ and a superblock $C$ of $\mathcal{B}$, how to compute

$$Refine(\mathcal{B}, C) = \bigcup_{B \in \mathcal{B}} Refine(B, C)$$

efficiently **?**

where for all blocks $B \in \mathcal{B}$:

$$Refine(B, C) = \{\, B \cap Pre(C),\ B \setminus Pre(C) \,\} \setminus \{\varnothing\}$$



block $B$ superblock $C$

# Refinement operator $Refine(\mathcal{B}, C)$

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```
FOR ALL s' ∈ C DO



OD
```

# Refinement operator $Refine(\mathcal{B}, C)$



```
FOR ALL s' ∈ C DO
   FOR ALL s ∈ Pre(s') DO



   OD
OD
```

# Refinement operator $Refine(\mathcal{B}, C)$



```
FOR ALL s' ∈ C DO
   FOR ALL s ∈ Pre(s') DO
         "move" state s from block [s]_B = B
                to the new block B ∩ Pre(C)
   OD
OD
```

## Refinement operator $Refine(\mathcal{B}, C)$

```
FOR ALL s′ ∈ C DO
   FOR ALL s ∈ Pre(s′) DO
         "move" state s from block [s]_B = B
                  to the new block B ∩ Pre(C)
   OD
OD
```

... states left in block $B \in \mathcal{B}$ belong to the new block $B \setminus Pre(C)$

```
FOR ALL s' ∈ C DO
   FOR ALL s ∈ Pre(s') DO
        "move" state s from block [s]_B = B
              to the new block B ∩ Pre(C)
   OD
OD
```

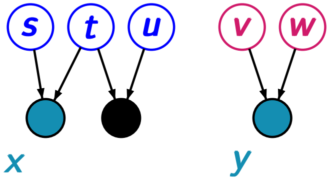
... states left in block $B \in \mathcal{B}$ belong to the new block $B \setminus \mathit{Pre}(C)$

time complexity:

$$\mathcal{O}\Big( \sum_{s' \in C} |\mathit{Pre}(s')| + |C| \Big)$$

# Example: refinement operator

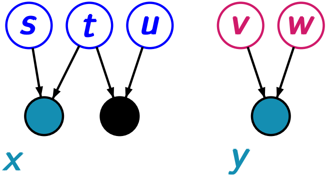





partition $\mathcal{B}$

# Example: refinement operator

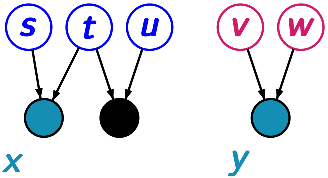





superblock $C = \{x, y\}$

partition $\mathcal{B} \rightsquigarrow \mathit{Refine}(\mathcal{B}, C)$

# Example: refinement operator

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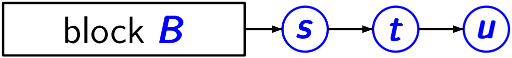
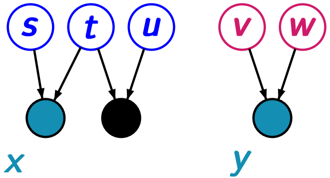




...

# Example: refinement operator



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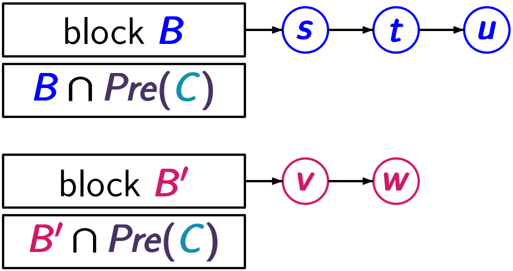
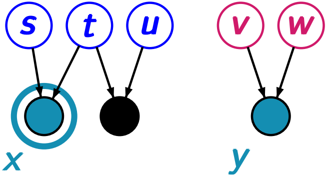



...

# Example: refinement operator




superblock $C = \{x, y\}$

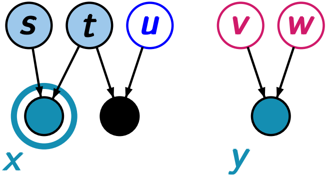
partition $\mathcal{B} \rightsquigarrow Refine(\mathcal{B}, C)$

| block $B$ | $\rightarrow s \rightarrow t \rightarrow u$ |
|---|---|
| $B \cap Pre(C)$ | |

| block $B'$ | $\rightarrow v \rightarrow w$ |
|---|---|
| $B' \cap Pre(C)$ | |

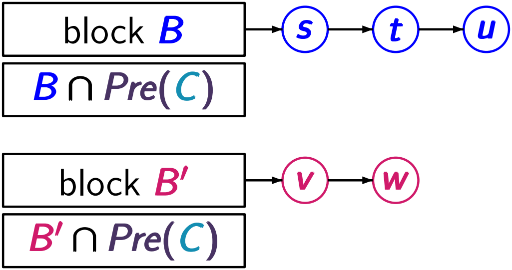
...

# Example: refinement operator




superblock $C = \{x, y\}$

partition $\mathcal{B} \rightsquigarrow \mathit{Refine}(\mathcal{B}, C)$




...

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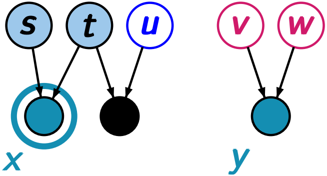


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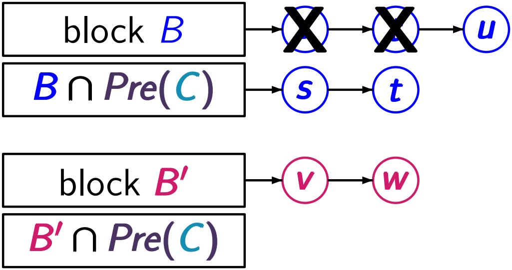
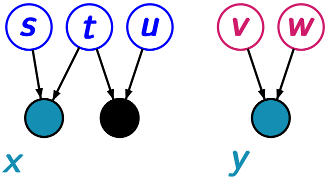



...

# Example: refinement operator






superblock $C = \{x, y\}$

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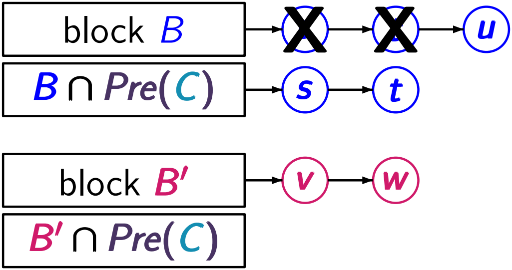
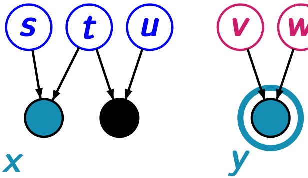



$\longleftarrow$ new block $B \setminus Pre(C)$

...

# Example: refinement operator

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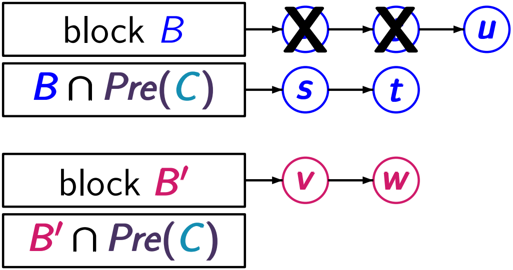


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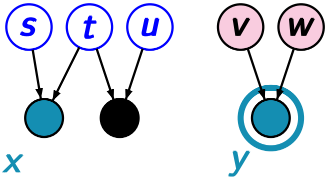


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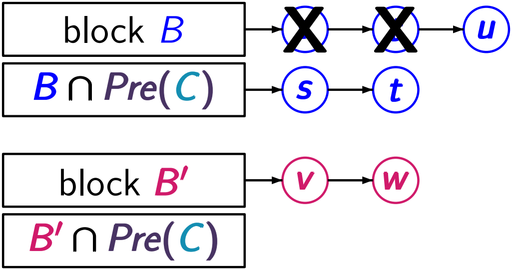


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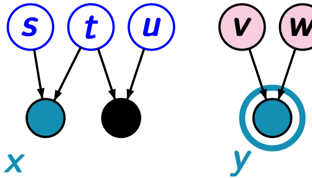


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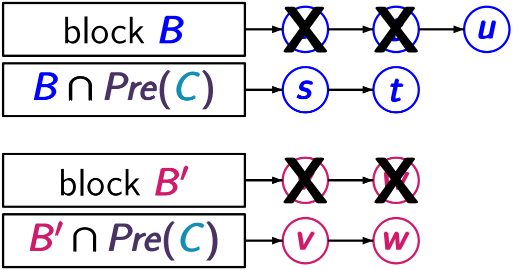
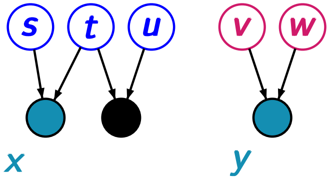



$\leftarrow$ new block $B \setminus \mathit{Pre}(C)$

$\ldots$

# Example: refinement operator



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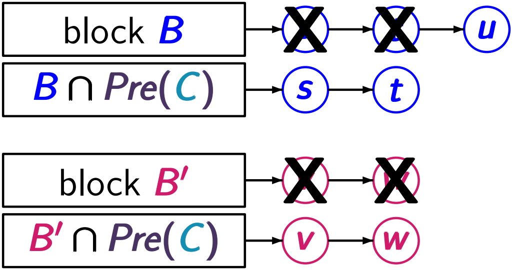
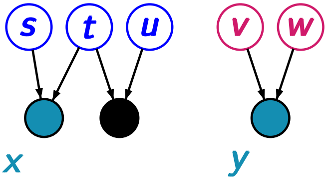



$\leftarrow$ new block $B \setminus \mathit{Pre}(C)$

$\leftarrow$ $B' \setminus \mathit{Pre}(C) = \varnothing$

...

# Example: refinement operator

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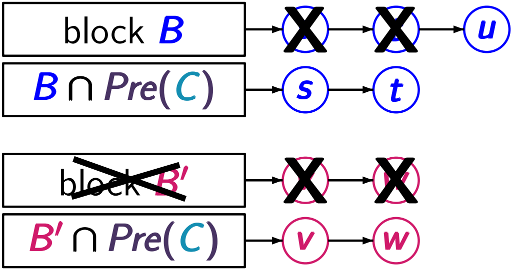


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...

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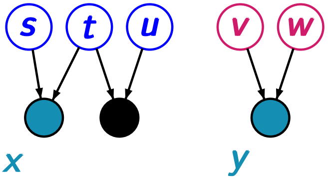
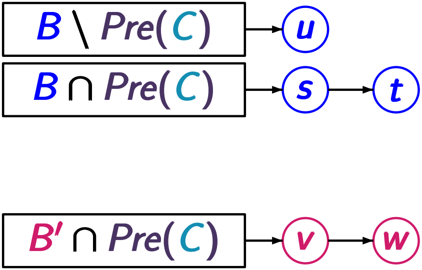



superblock $C = \{x, y\}$

$Refine(\mathcal{B}, C)$




...

# Partitioning splitter algorithm



$\mathcal{B} := \mathcal{B}_{AP}$

`WHILE` there is a splitter $C$ for $\mathcal{B}$ `DO`

  select such a splitter $C$;
  $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C)$

`OD`

return $\mathcal{B}$

# Partitioning splitter algorithm



$\mathcal{B} := \mathcal{B}_{AP}$ $\longleftarrow$ $\boxed{\text{time complexity: } \mathcal{O}(|S| \cdot |AP|)}$

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time complexity:

$$\mathcal{O}\Big( \sum_{C} \big( \sum_{s' \in C} |\mathit{Pre}(s')| + |C| \big) \; + \; |S| \cdot |AP| \Big)$$

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time complexity:

$$\mathcal{O}\Big( \sum_{C} \big( \sum_{s' \in C} |Pre(s')| + |C| \big) \ + \ |S| \cdot |AP| \Big)$$

$+$ cost for splitter search and management

# Partitioning splitter algorithms



**2** instances of the partitioning splitter algorithm
that differ in the choice and management of splitters

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- *Kanellakis-Smolka algorithm:*

  refinement according to all blocks of the partition of the previous iteration

## Partitioning splitter algorithms

2 instances of the partitioning splitter algorithm that differ in the choice and management of splitters

- *Kanellakis-Smolka algorithm:*

  refinement according to all blocks of the partition of the previous iteration

- *Paige-Tarjan-algorithm:*

  simultaneous refinement according to 2 superblocks

# Kanellakis-Smolka algorithm

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\text{old}} := \{S\}$

REPEAT

    $\mathcal{B}_{\text{old}} := \mathcal{B}$;

    FOR ALL $C \in \mathcal{B}_{\text{old}}$ DO $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C)$ OD

UNTIL $\mathcal{B} = \mathcal{B}_{\text{old}}$

return $\mathcal{B}$

# Kanellakis-Smolka algorithm



$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\mathsf{old}} := \{S\}$ $\longleftarrow$ $\boxed{\text{cost: } \mathcal{O}(|S| \cdot |AP|)}$

```
REPEAT
    B_old := B;
    FOR ALL C ∈ B_old DO B := Refine(B, C) OD
UNTIL B = B_old
return B
```

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\text{old}} := \{S\}$ $\longleftarrow$ $\boxed{\text{cost: } \mathcal{O}(|S|\cdot|AP|)}$

```
REPEAT
    B_old := B;
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```

- maximal $|S|$ iterations

# Kanellakis-Smolka algorithm



$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\text{old}} := \{S\}$ $\longleftarrow$ $\boxed{\text{cost: } \mathcal{O}(|S| \cdot |AP|)}$

REPEAT

  $\mathcal{B}_{\text{old}} := \mathcal{B}$;

  FOR ALL $C \in \mathcal{B}_{\text{old}}$ DO $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C)$ OD

UNTIL $\mathcal{B} = \mathcal{B}_{\text{old}}$

return $\mathcal{B}$

- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs

$$\mathcal{O}(|\mathit{Pre}(s')| + 1)$$

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{old} := \{S\}$ $\longleftarrow$ $\boxed{\text{cost: } \mathcal{O}(|S| \cdot |AP|)}$

```
REPEAT
    B_old := B;
    FOR ALL C ∈ B_old DO B := Refine(B, C) OD
UNTIL B = B_old
return B
```

$$\begin{array}{l} \texttt{REPEAT} \\ \quad \mathcal{B}_{old} := \mathcal{B}; \\ \quad \texttt{FOR ALL } C \in \mathcal{B}_{old} \texttt{ DO } \mathcal{B} := \mathit{Refine}(\mathcal{B}, C) \texttt{ OD} \\ \texttt{UNTIL } \mathcal{B} = \mathcal{B}_{old} \\ \text{return } \mathcal{B} \end{array}$$

- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs
  $$\mathcal{O}(|Pre(s')| + 1)$$
- cost per iteration: $\mathcal{O}(m + |S|)$

## Kanellakis-Smolka algorithm

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\text{old}} := \{S\}$ $\longleftarrow$ $\boxed{\text{cost: } \mathcal{O}(|S| \cdot |AP|)}$

REPEAT

  $\mathcal{B}_{\text{old}} := \mathcal{B}$;

  FOR ALL $C \in \mathcal{B}_{\text{old}}$ DO $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C)$ OD

UNTIL $\mathcal{B} = \mathcal{B}_{\text{old}}$

return $\mathcal{B}$

- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs
  $$\mathcal{O}(|\mathit{Pre}(s')| + 1)$$
- cost per iteration: $\mathcal{O}(m + |S|)$
  if $m$ = number of edges $= \sum_{s'} |\mathit{Pre}(s')|$

# Kanellakis-Smolka algorithm

```
B := B_AP;  B_old := {S}        <-- cost: O(|S|·|AP|)
REPEAT
    B_old := B;
    FOR ALL  C ∈ B_old DO  B := Refine(B, C) OD
UNTIL  B = B_old
return B
```

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\text{old}} := \{S\}$ $\longleftarrow$ $\boxed{\text{cost: } \mathcal{O}(|S|\cdot|AP|)}$

REPEAT

  $\mathcal{B}_{\text{old}} := \mathcal{B}$;

  FOR ALL $C \in \mathcal{B}_{\text{old}}$ DO $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C)$ OD

UNTIL $\mathcal{B} = \mathcal{B}_{\text{old}}$

return $\mathcal{B}$

- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs
  $$\mathcal{O}(|\mathit{Pre}(s')| + 1)$$
- cost per iteration: $\mathcal{O}(m + |S|) = \mathcal{O}(m)$
  if $m$ = number of edges $= \sum_{s'} |\mathit{Pre}(s')| \geq |S|$

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\mathsf{old}} := \{S\}$

REPEAT

$\quad\mathcal{B}_{\mathsf{old}} := \mathcal{B}$;

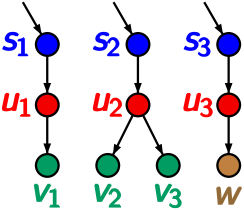
$\quad$FOR ALL $C \in \mathcal{B}_{\mathsf{old}}$ DO $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C)$ OD

UNTIL $\mathcal{B} = \mathcal{B}_{\mathsf{old}}$

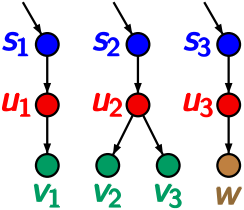
return $\mathcal{B}$

time complexity:
$\mathcal{O}(|S| \cdot m + |S| \cdot |AP|)$

- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs
$$\mathcal{O}(|Pre(s')| + 1)$$
- cost per iteration: $\mathcal{O}(m + |S|) = \mathcal{O}(m)$
  if $m$ = number of edges $= \sum_{s'} |Pre(s')| \geq |S|$

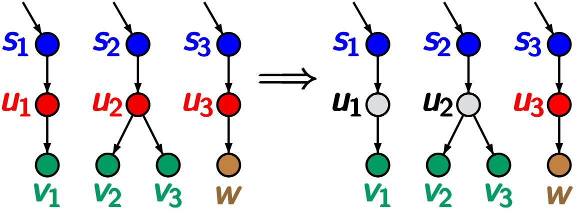

$s_1$
$s_2$
$s_3$
$u_1$
$u_2$
$u_3$
$v_1$
$v_2$
$v_3$
$w$

# Example: Kanellakis-Smolka algorithm




**1. iteration**:

1. refinement w.r.t. $\{v_1, v_2, v_3\}$

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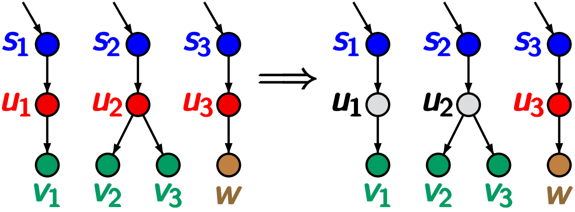
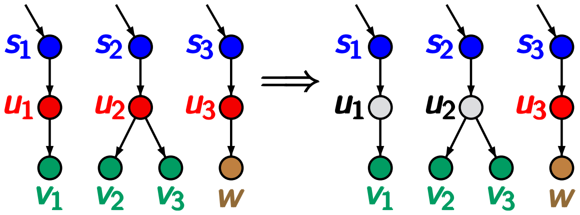



**1. iteration**:

1. refinement w.r.t. $\{v_1, v_2, v_3\}$
2. refinement w.r.t. $\{w\}$: no changes

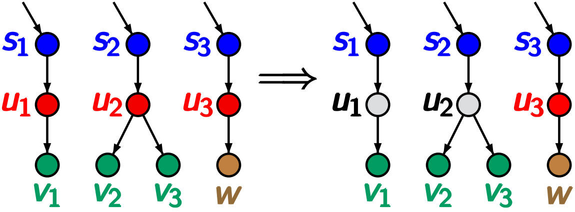
# Example: Kanellakis-Smolka algorithm




**1. iteration**:

1. refinement w.r.t. $\{v_1, v_2, v_3\}$
2. refinement w.r.t. $\{w\}$: no changes
3. refinement w.r.t. $\{s_1, s_2, s_3\}$: no changes

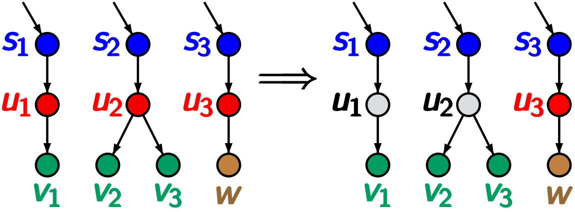
# Example: Kanellakis-Smolka algorithm




**1. iteration**:

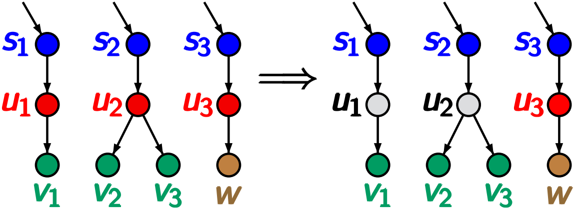
1. refinement w.r.t. $\{v_1, v_2, v_3\}$
2. refinement w.r.t. $\{w\}$: no changes
3. refinement w.r.t. $\{s_1, s_2, s_3\}$: no changes
4. refinement w.r.t. $\{u_1, u_2, u_3\}$: no changes

# Example: Kanellakis-Smolka algorithm




**2. iteration**:

# Example: Kanellakis-Smolka algorithm

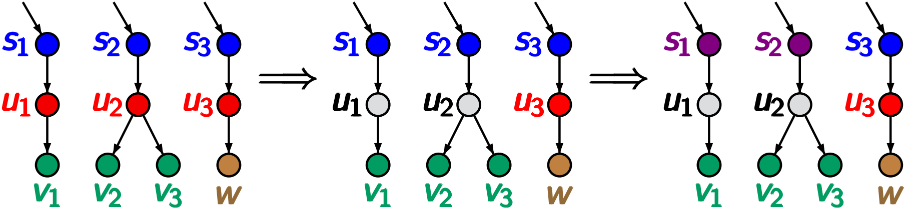



**2. iteration**:

1. refinement w.r.t. $\{u_3\}$

# Example: Kanellakis-Smolka algorithm

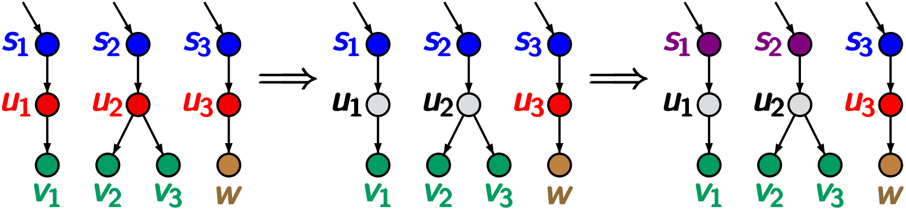





**2. iteration**:

1. refinement w.r.t. $\{u_3\}$

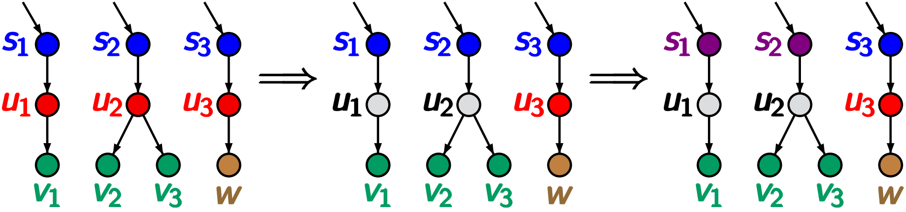
# Example: Kanellakis-Smolka algorithm




**2. iteration**:

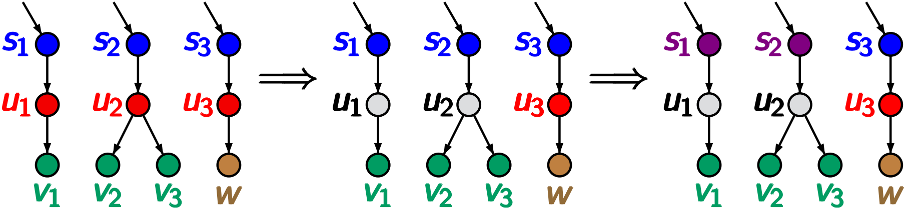
1. refinement w.r.t. $\{u_3\}$
2. refinement w.r.t. other blocks of the first iteration: no changes

# Example: Kanellakis-Smolka algorithm



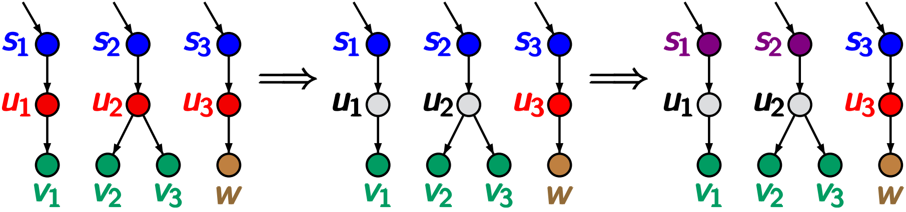

**3. iteration**:

# Example: Kanellakis-Smolka algorithm




**3. iteration**:

refinement w.r.t. all blocks of the second iteration: no changes




**3. iteration**:

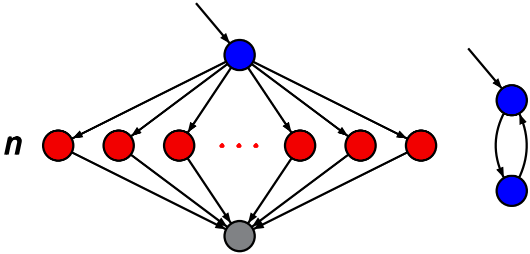
refinement w.r.t. all blocks of the second iteration: no changes

**6** bisimulation equivalence classes:

$\{s_1, s_2\}, \{s_3\}, \{u_1, u_2\}, \{u_3\}, \{v_1, v_2, v_3\}, \{w\}$

# Partitioning splitter algorithm






$AP = \{a, b\}$

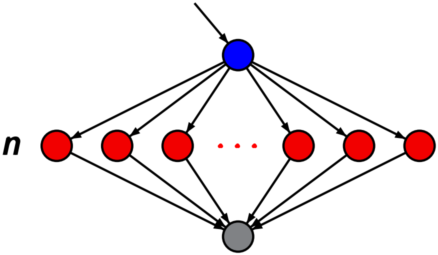
$\bullet \;\widehat{=}\; \{a\}$

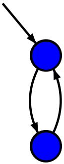
$\bullet \;\widehat{=}\; \{b\}$

$\bullet \;\widehat{=}\; \varnothing$

# Partitioning splitter algorithm





$AP = \{a, b\}$

● $\widehat{=}$ $\{a\}$

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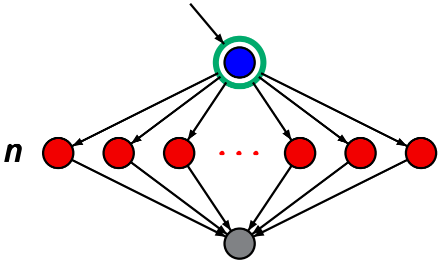
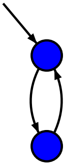
refinement w.r.t. ●:



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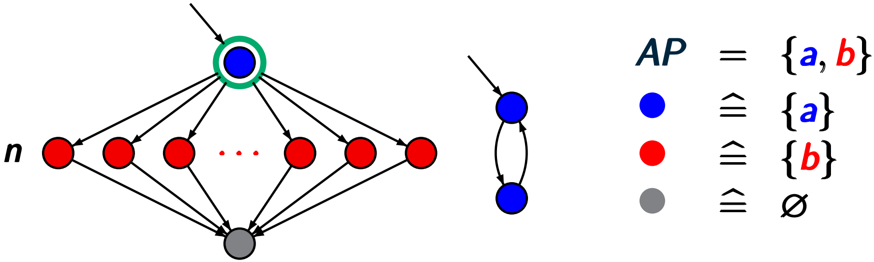
refinement w.r.t. ●:

# Partitioning splitter algorithm



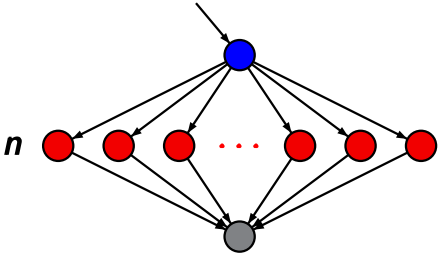


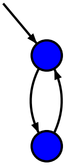

refinement w.r.t. ●: causes the costs

$$\sum_{s'} \left| Pre(s') \right| = n$$

# Partitioning splitter algorithm





$AP = \{a, b\}$

● $\widehat{=}$ $\{a\}$

● $\widehat{=}$ $\{b\}$

● $\widehat{=}$ $\varnothing$

refinement w.r.t. ●: causes the costs

$$\sum_{s'} |Pre(s')| = n$$

alternatively: refinement w.r.t. ●: constant costs

*Kanellakis-Smolka algorithm:*

initially: $\mathcal{B}_{\mathsf{old}} = \mathcal{B} = \mathcal{B}_{AP}$

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# Partitioning splitter algorithms

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*Paige-Tarjan algorithm:*

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*Paige-Tarjan algorithm:*

loop invariant:

(1) $\mathcal{B}$ finer than $\mathcal{B}_{\text{old}}$ and coarser than $\mathcal{B}_{AP}$

(2) $\mathcal{B}$ is stable for each block in $\mathcal{B}_{\text{old}}$

# Partitioning splitter algorithms



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iteration: ternary refinement operator

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loop invariant:

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iteration: ternary refinement operator

initially: $\mathcal{B}_{\mathsf{old}} = \{S\}$

# Partitioning splitter algorithms



*Kanellakis-Smolka algorithm:*

initially: $\mathcal{B}_{\mathsf{old}} = \mathcal{B} = \mathcal{B}_{AP}$

iteration: stabilization for each block in $\mathcal{B}_{\mathsf{old}}$

loop invariant: $\mathcal{B}$ finer than $\mathcal{B}_{\mathsf{old}}$ and coarser than $\mathcal{B}_{AP}$

*Paige-Tarjan algorithm:*

loop invariant:

(1) $\mathcal{B}$ finer than $\mathcal{B}_{\mathsf{old}}$ and coarser than $\mathcal{B}_{AP}$

(2) $\mathcal{B}$ is stable for each block in $\mathcal{B}_{\mathsf{old}}$

iteration: ternary refinement operator

initially: $\mathcal{B}_{\mathsf{old}} = \{S\}$, $\mathcal{B} = \mathit{Refine}(\mathcal{B}_{AP}, S)$

# $\mathcal{B}_{AP}$ is generally not stable w.r.t. $S$

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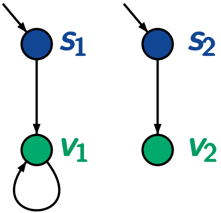
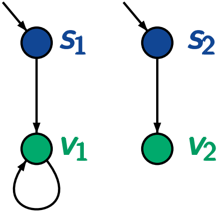


# $\mathcal{B}_{AP}$ is generally not stable w.r.t. $S$

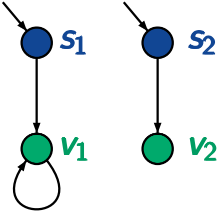



state space $S = \{s_1, s_2, v_1, v_2\}$

$$\mathcal{B}_{AP} = \Big\{ \{s_1, s_2\}, \{v_1, v_2\} \Big\}$$

# $\mathcal{B}_{AP}$ is generally not stable w.r.t. $S$



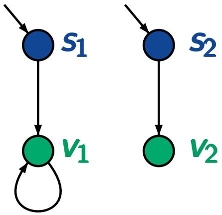



state space $S = \{s_1, s_2, v_1, v_2\}$

$$\mathcal{B}_{AP} = \Big\{ \{s_1, s_2\}, \{v_1, v_2\} \Big\}$$

$$\begin{aligned} Pre(S) &= \text{set of nonterminal states} \\ &= \{s_1, s_2, v_1\} \end{aligned}$$

# $\mathcal{B}_{AP}$ is generally not stable w.r.t. $S$




state space $S = \{s_1, s_2, v_1, v_2\}$

$\mathcal{B}_{AP} = \{\{s_1, s_2\}, \{v_1, v_2\}\}$

$Pre(S) =$ set of nonterminal states

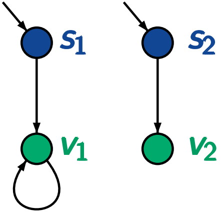
$= \{s_1, s_2, v_1\}$

$$\{v_1, v_2\} \cap Pre(S) = \{v_1\}$$

$$\{v_1, v_2\} \setminus Pre(S) = \{v_2\}$$

# $\mathcal{B}_{AP}$ is generally not stable w.r.t. $S$






state space $S = \{s_1, s_2, v_1, v_2\}$

$\mathcal{B}_{AP} = \big\{ \{s_1, s_2\}, \{v_1, v_2\} \big\}$

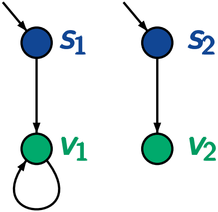
$Pre(S) =$ set of nonterminal states $= \{s_1, s_2, v_1\}$

$$\{v_1, v_2\} \cap Pre(S) = \{v_1\}$$
$$\{v_1, v_2\} \setminus Pre(S) = \{v_2\}$$

initial partition of Paige/Tarjan algorithm:

$$Refine(\mathcal{B}_{AP}, S)$$

# $\mathcal{B}_{AP}$ is generally not stable w.r.t. $S$




state space $S = \{s_1, s_2, v_1, v_2\}$

$$\mathcal{B}_{AP} = \Big\{ \{s_1, s_2\}, \{v_1, v_2\} \Big\}$$

$Pre(S)$ = set of nonterminal states $= \{s_1, s_2, v_1\}$

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initial partition of Paige/Tarjan algorithm:

$$Refine(\mathcal{B}_{AP}, S) = \Big\{ \{s_1, s_2\}, \{v_1\}, \{v_2\} \Big\}$$

# Paige-Tarjan algorithm

$\mathcal{B}_{\mathsf{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

`WHILE` $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ `DO`

`OD`

$\mathcal{B}_{\mathsf{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

```
WHILE B ≠ B_old DO
    select a block C' ∈ B_old \ B;
OD
```

$\mathcal{B}_{\mathsf{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO

select a block $C' \in \mathcal{B}_{\mathsf{old}} \setminus \mathcal{B}$;

select a block $C \in \mathcal{B}$ with $C \subseteq C'$

OD

$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

WHILE $\mathcal{B} \neq \mathcal{B}_{\text{old}}$ DO

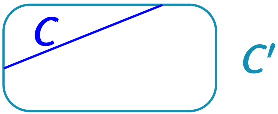
select a block $C' \in \mathcal{B}_{\text{old}} \setminus \mathcal{B}$;

select a block $C \in \mathcal{B}$ with $C \subseteq C'$ and $|C| \leq |C'|/2$;

OD

# Paige-Tarjan algorithm



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```
WHILE B ≠ B_old DO
   select a block C' ∈ B_old \ B;
   select a block C ∈ B with C ⊆ C' and |C| ≤ |C'|/2;

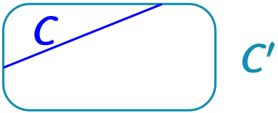





OD
```

# Paige-Tarjan algorithm






$\mathcal{B}_{\mathsf{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO

  select a block $C' \in \mathcal{B}_{\mathsf{old}} \setminus \mathcal{B}$;

  select a block $C \in \mathcal{B}$ with $C \subseteq C'$ and $|C| \le |C'|/2$;

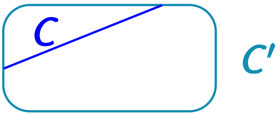
  refine $\mathcal{B}$
  w.r.t. $C$ and $C' \setminus C$

OD

# Paige-Tarjan algorithm



$\mathcal{B}_{\mathsf{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;




WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO

select a block $C' \in \mathcal{B}_{\mathsf{old}} \setminus \mathcal{B}$;

select a block $C \in \mathcal{B}$ with $C \subseteq C'$ and $|C| \leq |C'|/2$;

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C)$

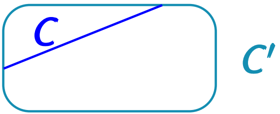
$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C')$

refine $\mathcal{B}$
w.r.t. $C$ and $C' \setminus C$

OD

# Paige-Tarjan algorithm






$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

```
WHILE 𝓑 ≠ 𝓑_old DO
```

select a block $C' \in \mathcal{B}_{\text{old}} \setminus \mathcal{B}$;

select a block $C \in \mathcal{B}$ with $C \subseteq C'$ and $|C| \leq |C'|/2$;

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C)$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C')$

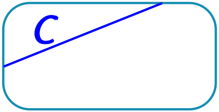
refine $\mathcal{B}$ simultaneously
w.r.t. $C$ and $C' \setminus C$

```
OD
```

# Paige-Tarjan algorithm



$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;




$C'$

```
WHILE
```
$\mathcal{B} \neq \mathcal{B}_{\text{old}}$ DO

select a block $C' \in \mathcal{B}_{\text{old}} \setminus \mathcal{B}$;

select a block $C \in \mathcal{B}$ with $C \subseteq C'$ and $|C| \leq |C'|/2$;

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

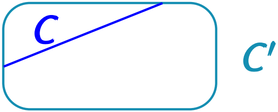
refine $\mathcal{B}$ simultaneously w.r.t. $C$ and $C' \setminus C$

OD

# Paige-Tarjan algorithm



$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;




```
WHILE  𝓑 ≠ 𝓑old  DO
```

select a block $C' \in \mathcal{B}_{\text{old}} \setminus \mathcal{B}$;

select a block $C \in \mathcal{B}$ with $C \subseteq C'$ and $|C| \leq |C'|/2$;

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

refine $\mathcal{B}$ simultaneously w.r.t. $C$ and $C' \setminus C$

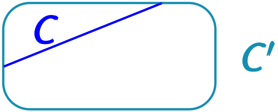
add $C$ and $C' \setminus C$ to $\mathcal{B}_{\text{old}}$

```
OD
```

# Paige-Tarjan algorithm



$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := \textit{Refine}(\mathcal{B}_{AP}, S)$;




WHILE $\mathcal{B} \neq \mathcal{B}_{\text{old}}$ DO

select a block $C' \in \mathcal{B}_{\text{old}} \setminus \mathcal{B}$;

select a block $C \in \mathcal{B}$ with $C \subseteq C'$ and $|C| \leq |C'|/2$;

$\mathcal{B} := \textit{Refine}(\mathcal{B}, C, C' \setminus C)$

refine $\mathcal{B}$ simultaneously w.r.t. $C$ and $C' \setminus C$

add $C$ and $C' \setminus C$ to $\mathcal{B}_{\text{old}}$ and remove $C'$ from $\mathcal{B}_{\text{old}}$

OD

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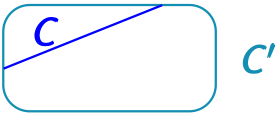


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OD

loop invariant: $\mathcal{B}$ is stable w.r.t. each block in $\mathcal{B}_{\mathsf{old}}$

## The ternary refinement operator

Let $\mathcal{B}$ be a partition and

- $C'$ a superblock of $\mathcal{B}$ s.t. $\mathcal{B}$ is stable w.r.t. $C'$

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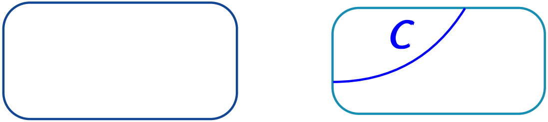


block $B$

superblock $C'$

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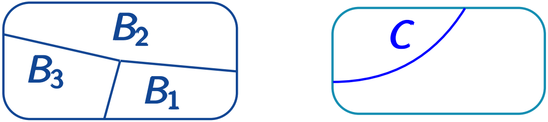


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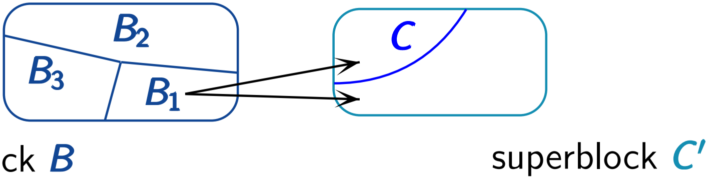


$$B_1 = B \cap Pre(C) \cap Pre(C' \setminus C)$$

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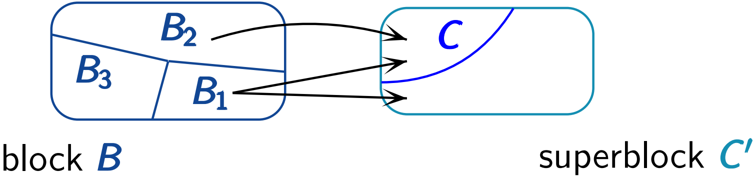


$$\begin{aligned} B_1 &= B \cap Pre(C) \cap Pre(C' \setminus C) \\ B_2 &= (B \cap Pre(C)) \setminus Pre(C' \setminus C) \end{aligned}$$

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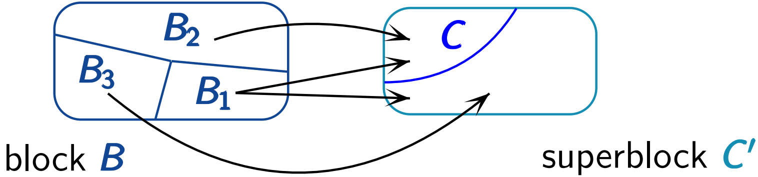


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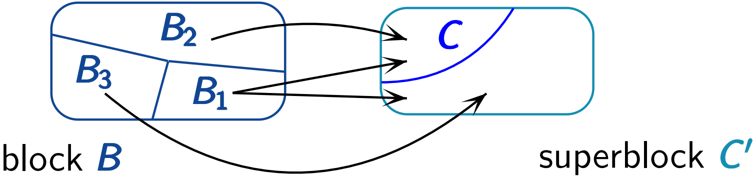


for block $B$ with $B \cap Pre(C') = \varnothing$:

$$Refine(B, C, C' \setminus C) = \{B\}$$

Suppose that for all blocks $B \in \mathcal{B}$:

$$B \subseteq Pre(C') \quad \text{or} \quad B \cap Pre(C') = \varnothing$$

# Stability of $\mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

Suppose that for all blocks $B \in \mathcal{B}$:

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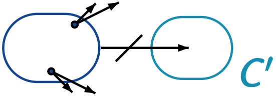
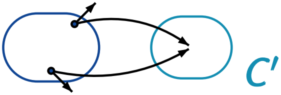



or



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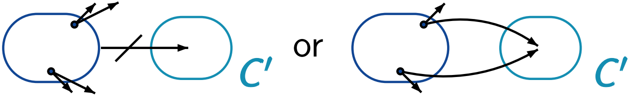
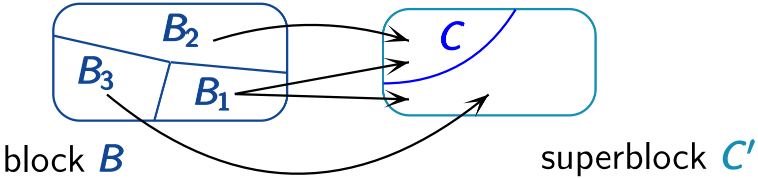



Then the new blocks $B_1, B_2, B_3$ in *Refine*$(B, C, C' \setminus C)$ are stable w.r.t. the superblocks $C$ and $C' \setminus C$.



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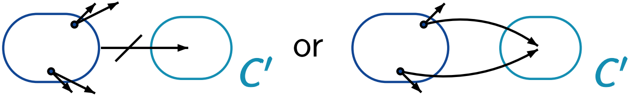


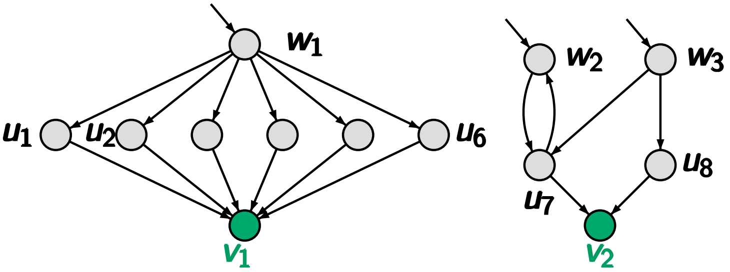

Then the new blocks $B_1, B_2, B_3$ in $Refine(B, C, C' \setminus C)$ are stable w.r.t. the superblocks $C$ and $C' \setminus C$.

If $\mathcal{B}$ is stable w.r.t. all blocks in $\mathcal{B}_{old}$ and $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t. $C \subsetneq C'$ then $Refine(\mathcal{B}, C, C' \setminus C)$ is stable w.r.t. all blocks in the partition

$$(\mathcal{B}_{old} \setminus \{C'\}) \cup \{C, C' \setminus C\}$$

# Example: Paige-Tarjan algorithm






$AP = \{green, gray\}, \quad \mathcal{B}_{old} = \{S\}$

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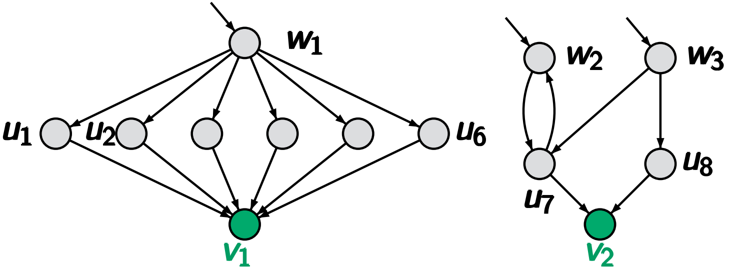


$AP = \{green, gray\}, \quad \mathcal{B}_{old} = \{S\}$

initial partition:

$$\begin{aligned}\mathcal{B}_0 &= Refine(\mathcal{B}_{AP}, S) = \mathcal{B}_{AP} \\ &= \big\{\{v_1, v_2\}, \{u_1, \ldots, u_8, w_1, w_2, w_3\}\big\}\end{aligned}$$

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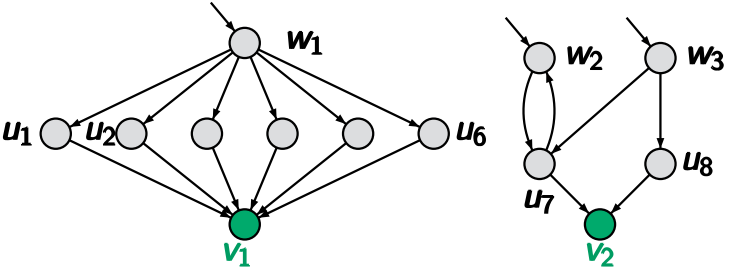


initially: $\mathcal{B}_{\text{old}} = \{S\}$

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first refinement step:

$Refine(\mathcal{B}_0, \{v_1, v_2\}, S \setminus \{v_1, v_2\})$

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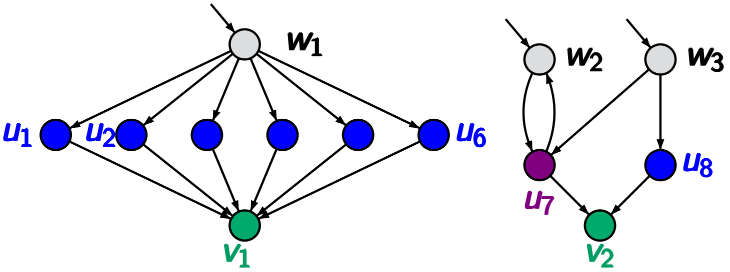


initially: $\mathcal{B}_{\text{old}} = \{S\}$

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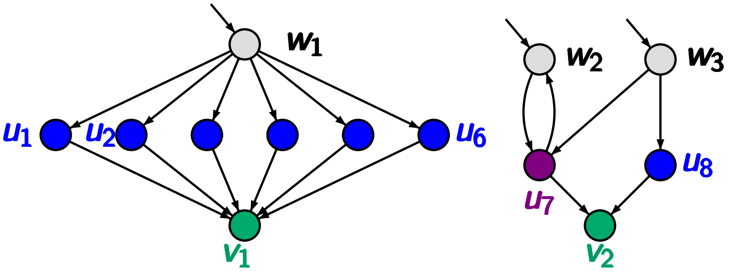


initially: $\mathcal{B}_{old} = \{S\}$

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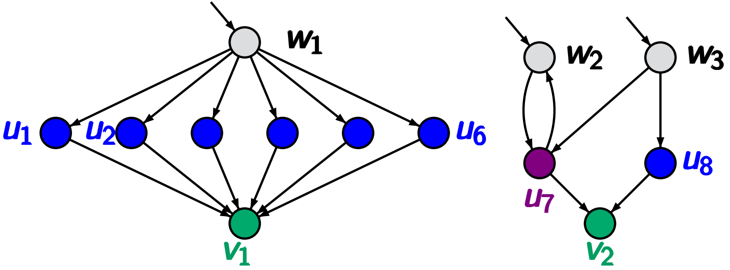


first refinement step:

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second refinement step:

$$\mathit{Refine}(\mathcal{B}_1, ?, ?)$$

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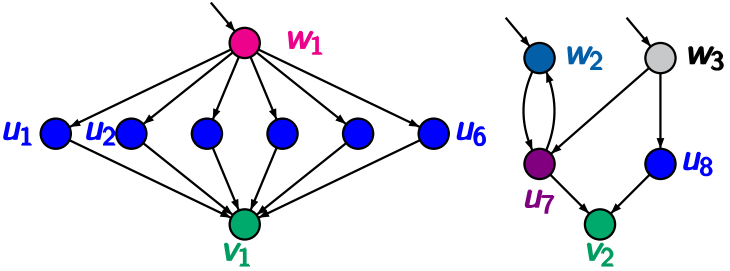


first refinement step:

$$\mathcal{B}_1 = \bigl\{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8\}, \{u_7\}, \{w_1, w_2, w_3\}\bigr\}$$
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second refinement step:

$$Refine(\mathcal{B}_1, \{u_7\}, \{u_1, \ldots, u_6, u_8, w_1, w_2, w_3\})$$

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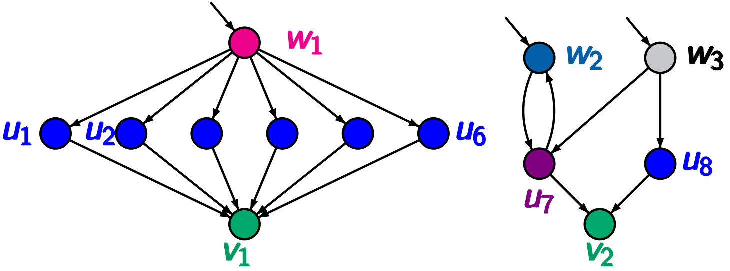


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second refinement step:

$$Refine(\mathcal{B}_1, \{u_7\}, \{u_1, \ldots, u_6, u_8, w_1, w_2, w_3\})$$
$$= \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8\}, \{u_7\}, \{w_1\}, \{w_2\}, \{w_3\}\}$$

# Paige-Tarjan algorithm



$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\mathcal{B}_{\mathsf{old}} := \{S\}$;

`WHILE` $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ `DO`

  select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

  add $C$ and $C' \setminus C$ to $\mathcal{B}_{\mathsf{old}}$ and remove $C'$ from $\mathcal{B}_{\mathsf{old}}$

  $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

`OD`

return $\mathcal{B}$

# Paige-Tarjan algorithm

```
B := Refine(B_AP, S);  B_old := {S};
WHILE B ≠ B_old DO
   select C' ∈ B_old, C ∈ B s.t. C ⊆ C', |C| ≤ |C'|/2;
   add C and C' \ C to B_old and remove C' from B_old
   B := Refine(B, C, C' \ C)
OD
return B
```

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\mathcal{B}_{\mathsf{old}} := \{S\}$;
WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO
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  $\boxed{\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)}$
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efficient implementation of $\mathit{Refine}(\mathcal{B}, C, \ldots)$ with time complexity $\mathcal{O}\big(|C| + |\mathit{Pre}(C)|\big)$

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OD
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efficient implementation of $\mathit{Refine}(\mathcal{B}, C, \ldots)$ with time complexity $\mathcal{O}\big(|C| + |\mathit{Pre}(C)|\big)$ uses counters

$$\delta(s, D) = |\mathit{Post}(s) \cap D| \text{ for } D \in \mathcal{B}_{\mathsf{old}}$$

implementation of

$$\mathit{Refine}(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} \mathit{Refine}(B, C, C' \setminus C)$$

using counters $\delta(s, D) = |\mathit{Post}(s) \cap D|$

# Details of the Paige-Tarjan algorithm

implementation of

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

using counters $\delta(s, D) = |Post(s) \cap D|$

$D \in \mathcal{B}_{\mathsf{old}}$

# Details of the Paige-Tarjan algorithm

implementation of

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

using counters $\delta(s, D) = |Post(s) \cap D|$

$s \in Pre(D)$ $\quad$ $D \in \mathcal{B}_{\text{old}}$

implementation of

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

using counters $\delta(s, D) = |Post(s) \cap D|$

$s \in Pre(D)$ $\qquad$ $D \in \mathcal{B}_{\text{old}}$

*step 1:* compute $\delta(\ldots)$ for the new blocks $C$ and $C' \setminus C$ in $\mathcal{B}_{\text{old}}$

implementation of

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

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$s \in Pre(D)$ $\quad$ $D \in \mathcal{B}_{\text{old}}$

*step 1:* compute $\delta(\ldots)$ for the new blocks $C$ and $C' \setminus C$ in $\mathcal{B}_{\text{old}}$

*step 2*: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

# Implementation of $\mathit{Refine}(\mathcal{B}, C, C' \setminus C)$



*step 1:* compute $\delta(s, C)$, $\delta(s, C' \setminus C)$

*step 2*: compute $\mathit{Refine}(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

# Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$ 

*step 1:* compute $\delta(s, C)$, $\delta(s, C' \setminus C) \longleftarrow \boxed{\text{for } s \in Pre(C')}$

*step 2*: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

# Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$



*step 1:* compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ $\boxed{\text{for } s \in Pre(C')}$

$$\delta(s, C) \qquad = |Post(s) \cap C|$$

$$\delta(s, C' \setminus C) = |Post(s) \cap (C' \setminus C)|$$

*step 2*: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

# Implementation of $\mathit{Refine}(\mathcal{B}, C, C' \setminus C)$



*step 1:* compute $\delta(s, C)$, $\delta(s, C' \setminus C) \longleftarrow \boxed{\text{for } s \in \mathit{Pre}(C')}$

$$\delta(s, C) = |\mathit{Post}(s) \cap C|$$

$$\delta(s, C' \setminus C) = |\mathit{Post}(s) \cap (C' \setminus C)|$$

*step 2:* compute $\mathit{Refine}(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

for $B \in \mathcal{B}$ with $B \cap \mathit{Pre}(C') = \varnothing$ we have:

$$\mathit{Refine}(B, C, C' \setminus C) = \{B\}$$

# Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$



*step 1:* compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ $\boxed{\text{for } s \in Pre(C')}$

$$\delta(s, C) \quad = |Post(s) \cap C|$$
$$\delta(s, C' \setminus C) = |Post(s) \cap (C' \setminus C)|$$

*step 2:* compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

for $B \in \mathcal{B}$ with $B \subseteq Pre(C')$:

# Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$



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$$B_1 = B \cap Pre(C) \cap Pre(C' \setminus C)$$

$$B_2 = (B \cap Pre(C)) \setminus Pre(C' \setminus C)$$

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for $B \in \mathcal{B}$ with $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$

$$B_1 = \{s \in B : \delta(s, C) > 0, \delta(s, C' \setminus C) > 0\}$$

$$B_2 = (B \cap Pre(C)) \setminus Pre(C' \setminus C)$$

$$B_3 = (B \cap Pre(C' \setminus C)) \setminus Pre(C)$$

# Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$



*step 1:* compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ $\boxed{\text{for } s \in Pre(C')}$

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$$B_2 = \{s \in B : \delta(s, C) > 0, \delta(s, C' \setminus C) = 0\}$$

$$B_3 = (B \cap Pre(C' \setminus C)) \setminus Pre(C)$$

# Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$



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$$B_3 = \{s \in B : \delta(s, C) = 0, \delta(s, C' \setminus C) > 0\}$$

# Paige-Tarjan algorithm

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{\mathsf{AP}}, S)$; $\mathcal{B}_{\mathsf{old}} := \{S\}$;

```
WHILE  B ≠ B_old DO
  select C' ∈ B_old, C ∈ B s.t. C ⊆ C', |C| ≤ |C'|/2;
  add C and C'\C to B_old and remove C' from B_old

  B := Refine(B, C, C' \ C)
OD
```

$\texttt{WHILE}\ \ \mathcal{B} \neq \mathcal{B}_{\mathsf{old}}\ \texttt{DO}$

select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

add $C$ and $C' \setminus C$ to $\mathcal{B}_{\mathsf{old}}$ and remove $C'$ from $\mathcal{B}_{\mathsf{old}}$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

$\texttt{OD}$

# Paige-Tarjan algorithm



$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{\mathsf{AP}}, S)$; $\mathcal{B}_{\mathsf{old}} := \{S\}$;

`FOR ALL` $s \in S$ `DO` $\delta(s, S) := |\mathit{Post}(s)|$ `OD`

`WHILE` $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ `DO`

select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

add $C$ and $C' \backslash C$ to $\mathcal{B}_{\mathsf{old}}$ and remove $C'$ from $\mathcal{B}_{\mathsf{old}}$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

`OD`

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{\mathrm{AP}}, S)$; $\mathcal{B}_{\mathrm{old}} := \{S\}$;

FOR ALL $s \in S$ DO $\delta(s, S) := |\mathit{Post}(s)|$ OD

WHILE $\mathcal{B} \neq \mathcal{B}_{\mathrm{old}}$ DO

select $C' \in \mathcal{B}_{\mathrm{old}}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

add $C$ and $C' \backslash C$ to $\mathcal{B}_{\mathrm{old}}$ and remove $C'$ from $\mathcal{B}_{\mathrm{old}}$

FOR ALL $s \in \mathit{Pre}(C)$ DO $\delta(s, C) := 0$ OD

FOR ALL $s' \in C$ DO

FOR ALL $s \in \mathit{Pre}(s')$ DO $\delta(s, C) := \delta(s, C) + 1$ OD

OD

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

OD

$\mathcal{B} := Refine(\mathcal{B}_{AP}, S)$; $\mathcal{B}_{old} := \{S\}$;

FOR ALL $s \in S$ DO $\delta(s, S) := |Post(s)|$ OD

WHILE $\mathcal{B} \neq \mathcal{B}_{old}$ DO

  select $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

  add $C$ and $C' \backslash C$ to $\mathcal{B}_{old}$ and remove $C'$ from $\mathcal{B}_{old}$

    FOR ALL $s \in Pre(C)$ DO $\delta(s, C) := \mathbf{0}$ OD

    FOR ALL $s' \in C$ DO

      FOR ALL $s \in Pre(s')$ DO $\delta(s, C) := \delta(s, C) + \mathbf{1}$ OD

    OD

    FOR ALL $s \in Pre(C)$ DO

      $\delta(s, C' \backslash C) := \delta(s, C') - \delta(s, C)$ OD

  $\mathcal{B} := Refine(\mathcal{B}, C, C' \setminus C)$

OD

# Complexity of the Paige-Tarjan algorithm PARTSPLITALG5.3-26A

# Complexity of the Paige-Tarjan algorithm 

let $\mathcal{T} = (S, Act, \to, S_0, AP, L)$ be a finite TS

$$n = \#\ \text{states} \qquad = |S|$$

$$m = \#\ \text{transitions}$$

let $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ be a finite TS

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# Complexity of the Paige-Tarjan algorithm 

let $\mathcal{T} = (S, Act, \to, S_0, AP, L)$ be a finite TS

$$n = \#\text{ states} = |S|$$
$$m = \#\text{ transitions} = \sum_{s \in S} |Pre(s)|$$

in what follows, we suppose $m \geq n$

# Complexity of the Paige-Tarjan algorithm

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{\mathsf{AP}}, S)$;

$\mathcal{B}_{\mathsf{old}} := \{S\}$;

`WHILE` $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ `DO`

select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t.
$C \subseteq C'$ and $|C| \leq |C'|/2$;

add $C$ and $C' \setminus C$ to $\mathcal{B}_{\mathsf{old}}$ and
remove $C'$ from $\mathcal{B}_{\mathsf{old}}$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

`OD`

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\longleftarrow$ complexity: $\mathcal{O}(n \cdot |AP|)$

$\mathcal{B}_{old} := \{S\}$;

```
WHILE
```
$\mathcal{B} \neq \mathcal{B}_{old}$ DO

select $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t.
$C \subseteq C'$ and $|C| \leq |C'|/2$;

add $C$ and $C' \setminus C$ to $\mathcal{B}_{old}$ and
remove $C'$ from $\mathcal{B}_{old}$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

OD

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\longleftarrow$ $\boxed{\text{complexity: } \mathcal{O}\big(n \cdot |AP|\big)}$

$\mathcal{B}_{old} := \{S\}$;

WHILE $\mathcal{B} \neq \mathcal{B}_{old}$ DO

select $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t.
$C \subseteq C'$ and $|C| \leq |C'|/2$;

add $C$ and $C' \backslash C$ to $\mathcal{B}_{old}$ and
remove $C'$ from $\mathcal{B}_{old}$

$$\boxed{\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)}$$

OD

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\longleftarrow$ complexity: $\mathcal{O}(n \cdot |AP|)$

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WHILE $\mathcal{B} \neq \mathcal{B}_{old}$ DO

select $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t.
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remove $C'$ from $\mathcal{B}_{old}$

$\boxed{\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)}$ $\longleftarrow$ time complexity: $\sum_{s' \in C} |\mathit{Pre}(s')| + 1$

OD

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{\mathsf{AP}}, S)$; $\longleftarrow$ complexity: $\mathcal{O}(n \cdot |AP|)$

$\mathcal{B}_{\mathsf{old}} := \{S\}$;

WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO

select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$ and $|C| \leq |C'|/2$;

add $C$ and $C' \setminus C$ to $\mathcal{B}_{\mathsf{old}}$ and remove $C'$ from $\mathcal{B}_{\mathsf{old}}$

$\boxed{\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)}$ $\longleftarrow$ time complexity: $\mathcal{O}(|C| + |\mathit{Pre}(C)|)$

OD

$\mathcal{B} := Refine(\mathcal{B}_{AP}, S)$; $\longleftarrow$ complexity: $\mathcal{O}(n \cdot |AP|)$

$\mathcal{B}_{old} := \{S\}$;

WHILE $\mathcal{B} \neq \mathcal{B}_{old}$ DO

select $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$ and $|C| \leq |C'|/2$;

add $C$ and $C' \setminus C$ to $\mathcal{B}_{old}$ and remove $C'$ from $\mathcal{B}_{old}$

$\boxed{\mathcal{B} := Refine(\mathcal{B}, C, C' \setminus C)}$ $\longleftarrow$ time complexity: $\mathcal{O}(|C| + |Pre(C)|)$

OD

total cost for all refinement operations: $\mathcal{O}(m \cdot \log n)$