Reachability in Markov Chains

Lecture #19 of Advanced Model Checking

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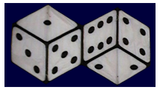
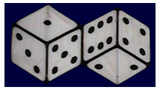


Probabilities help

- When analysing system performance and dependability
  - to quantify arrivals, waiting times, time between failure, QoS, ...
- When modelling uncertainty in the environment
  - to quantify environmental factors in decision support
  - to quantify unpredictable delays, express soft deadlines, ...
- When building protocols for networked embedded systems
  - randomized algorithms
- When analysing large populations
  - number of nodes in the internet, number of end-users, ...

Probabilistic verification so far

- Termination of probabilistic programs (Hart, Sharir & Pnueli, 1983)
  - does a probabilistic program terminate with probability one?
- Markov decision processes (Courcoubetis & Yannakakis, 1988)
  - does a certain (linear) temporal logic formula hold with probability p?
- Discrete-time Markov chains (Hansson & Jonsson, 1990)
  - can we reach a goal state via a given trajectory with probability p?
- Discrete-time Markov decision processes (Bianco & de Alfaro, 1995)
  - what is the maximal (or minimal) probability of doing this?
- Continuous-time Markov chains (Baier, Katoen & Hermanns, 1999)
  - can we do so within a given time interval I?



Characteristics

- What is inside?
  - temporal logics and model checking
  - numerical and optimisation techniques from performance and OR
- What can be checked?
  - time-bounded reachability, long-run averages, safety and liveness
- What is its usage?
  - powerful tools: PRISM (4,000 downloads), MRMC, Petri net tools, Probmela
  - applications: distributed systems, security, biology, quantum computing . . .

A synchronous leader election protocol

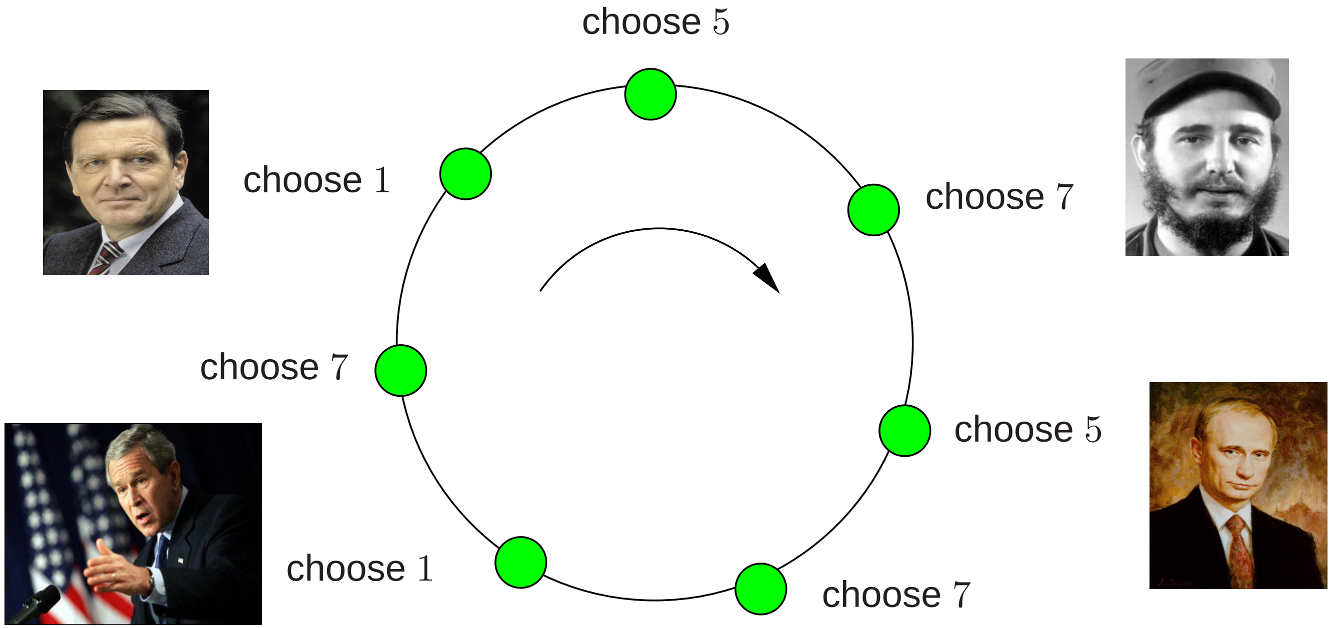
(Itai & Rodeh, 1990)

- A round-based protocol in a synchronous ring of $N > 2$ nodes
  - the nodes proceed in a lock-step fashion
  - each slot = 1 message is read + 1 state change + 1 message is sent

  $\Rightarrow$ this synchronous computation yields a Markov chain

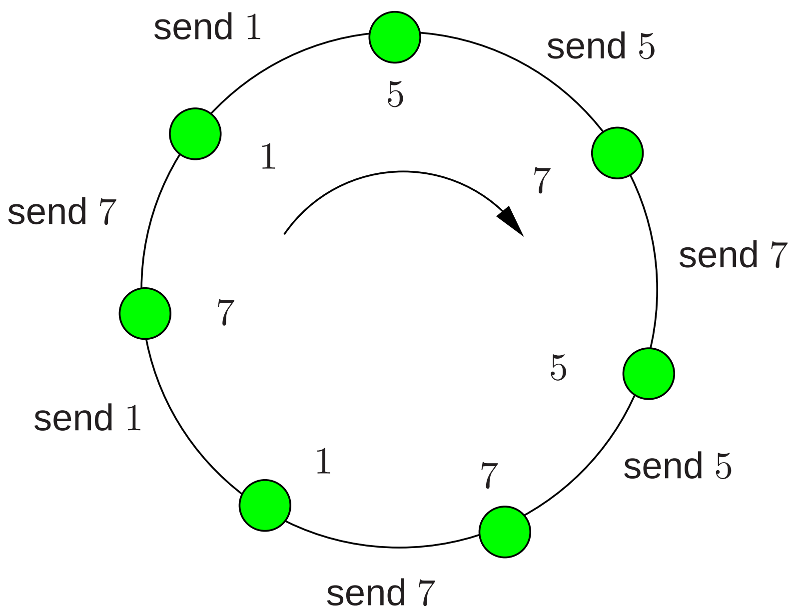
- Each round starts by each node choosing a uniform id $\in \{ 1, \ldots, K \}$
- Nodes pass their selected id around the ring
- If there is a unique id, the node with the maximum unique id is leader
- If not, start another round and try again $\ldots$

Leader election



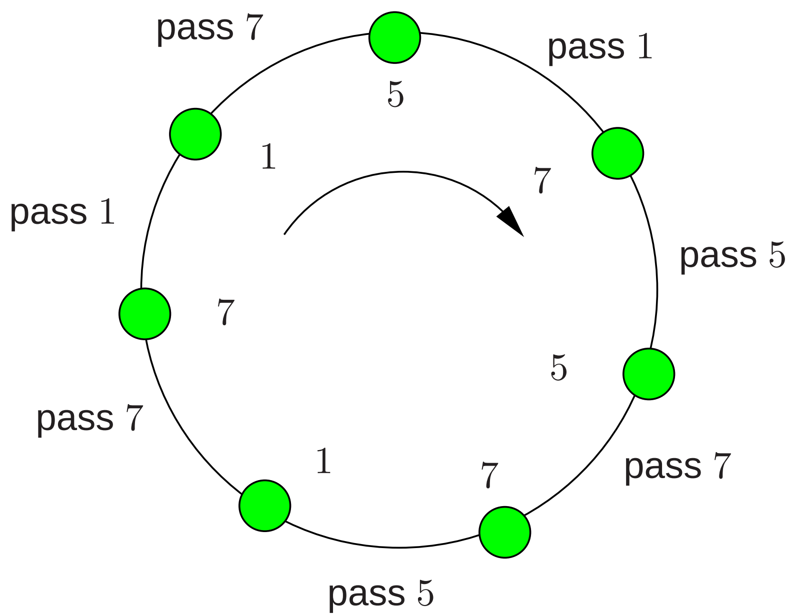
probabilistically choose an id from $[1 \ldots K]$

Leader election



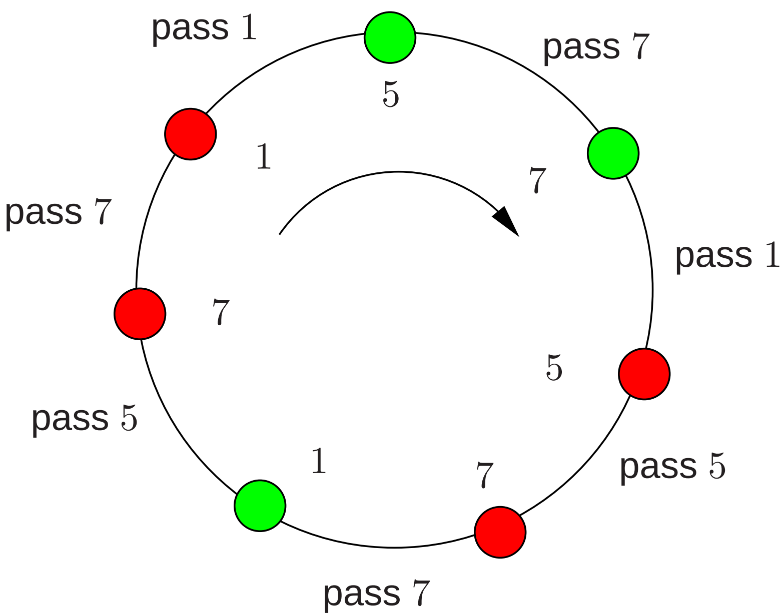
send your selected id to your neighbour

Leader election



pass the received id, and check uniqueness own id

Leader election



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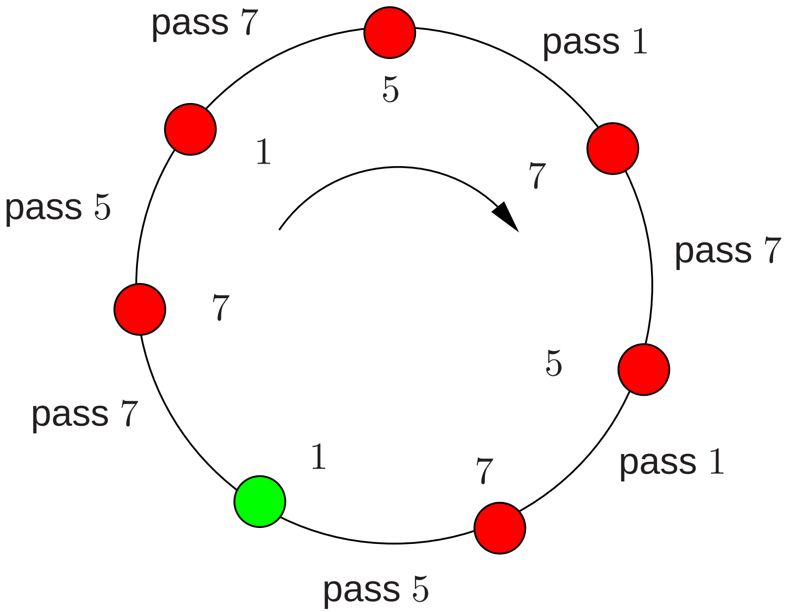
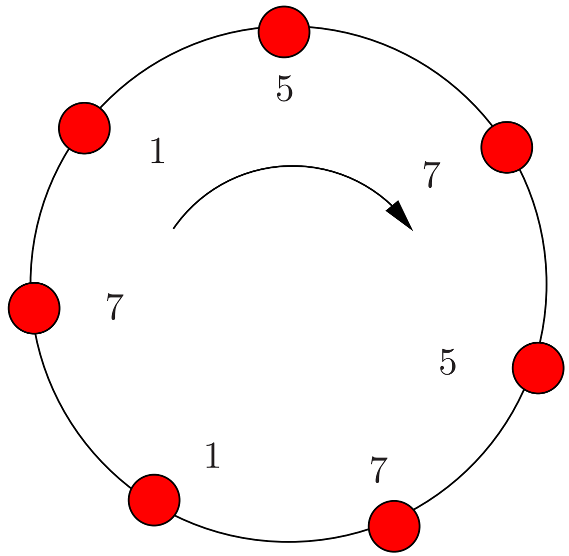
Leader election



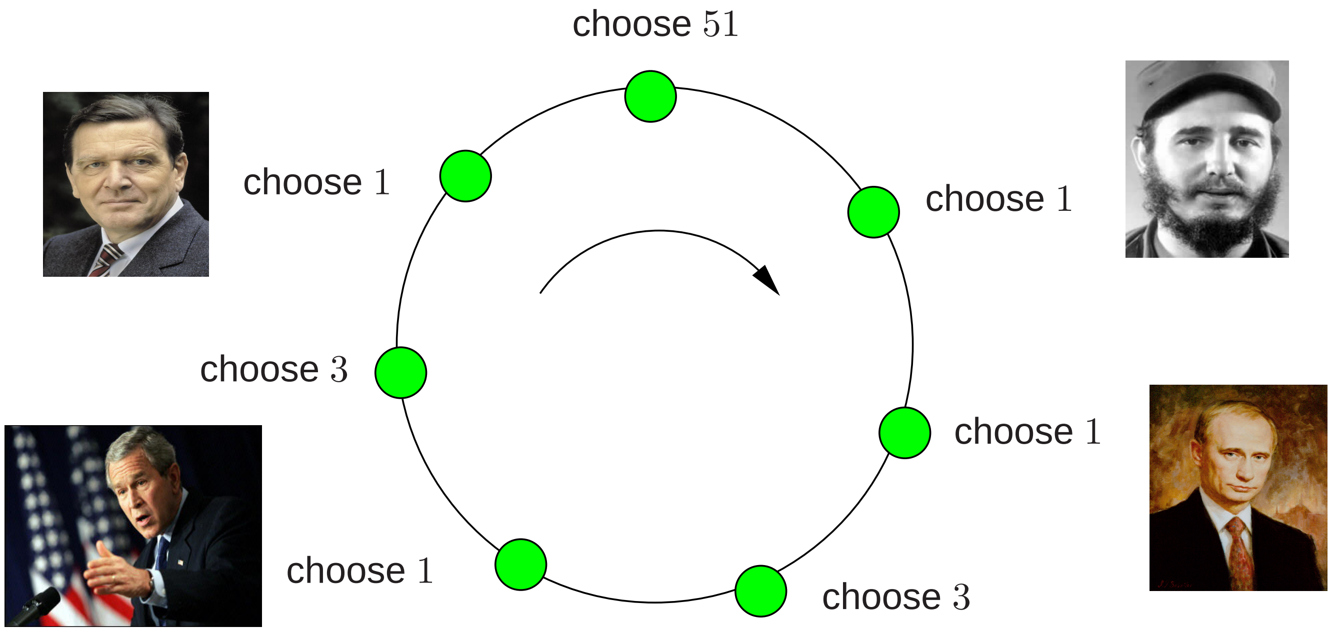
pass the received id, and check uniqueness own id

End of 1st round



no unique leader has been elected

Start a new round



new round and new chances!

Properties of leader election

- Almost surely eventually a leader will be elected:

$$\mathbb{P}_{=1}(\Diamond \textit{leader elected})$$

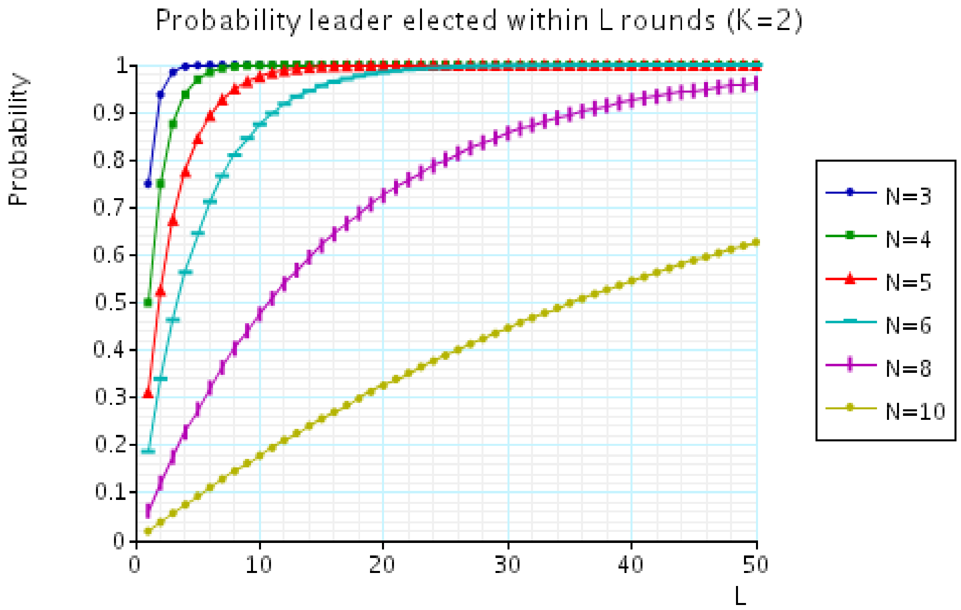
- With probability $\geqslant \frac{4}{5}$, eventually a leader is elected :

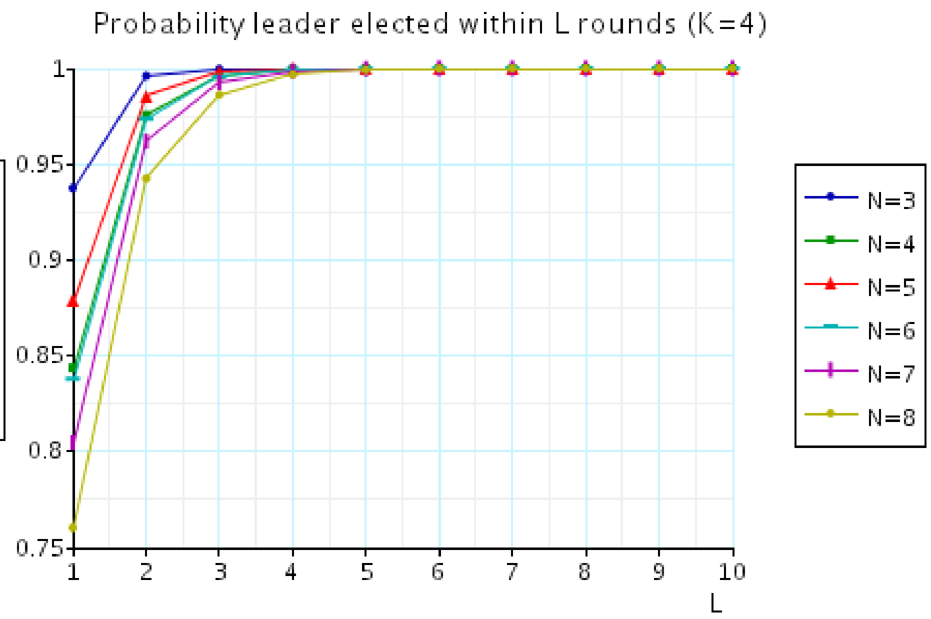
$$\mathbb{P}_{\geqslant 0.8}(\Diamond \textit{leader elected})$$

- . . . . . . within k steps:

$$\mathbb{P}_{\geqslant 0.8}(\Diamond^{\leqslant k} \textit{leader elected})$$

Probability to elect a leader within L rounds





$\mathbb{P}_{\leqslant q}(\Diamond^{\leqslant (N+1)\cdot L}$ *leader elected*$)$ (Itai & Rodeh's algorithm)

Discrete-time Markov chains

A DTMC $\mathcal{M}$ is a tuple $(S, \mathbf{P}, \iota_{init}, AP, L)$ with:

- S is a countable nonempty set of states
- $\mathbf{P} : S \times S \to [0,1]$, transition probability function s.t. $\sum_{s'} \mathbf{P}(s, s') = 1$
  - $\mathbf{P}(s, s')$ is the probability to jump from s to s' in one step
- $\iota_{init} : S \to [0,1]$, the initial distribution with $\sum_{s \in S} \iota_{init}(s) = 1$
  - $\iota_{init}(s)$ is the probability that system starts in state s
  - state s for which $\iota_{init}(s) > 0$ is an initial state
- $L : S \to 2^{AP}$, the labelling function

$\Rightarrow$ a DTMC is a transition system with only probabilistic transitions

Example

Paths

- State graph of DTMC $\mathcal{M}$
  - vertices are states of $\mathcal{M}$, and $(s, s') \in E$ if and only if $\mathbf{P}(s, s') > 0$
- Paths in $\mathcal{M}$ are maximal (i.e., infinite) paths inits state graph
  - for path π in $\mathcal{M}$, $\mathrm{inf}(\pi)$ is the set of states that are visited infinitely often in π
  - *Paths*$(\mathcal{M})$ and *Paths*$_{\mathit{fin}}(\mathcal{M})$ denote the set of (finite) paths in $\mathcal{M}$
- *Post*$(s) = \{ s' \in S \mid \mathbf{P}(s, s') > 0 \}$ and *Pre*$(s) = \{ s' \in S \mid \mathbf{P}(s', s) > 0 \}$
  - *Post*$^*(s)$ is the set of states reachable from s via a finite path fragment
  - *Pre*$^*(s) = \{ s' \in S \mid s \in$ *Post*$^*(s') \}$

σ-algebra

$(\Omega, \mathcal{F})$ with $\mathcal{F} \subseteq 2^{\Omega}$ is a *σ-algebra* if:

1. $\varnothing \in \mathcal{F}$
2. $E \in \mathcal{F} \;\Rightarrow\; \Omega - E \in \mathcal{F}$, and
3. $(\forall i \geqslant 0.\ E_i \in \mathcal{F})$ implies $\bigcup_{i \geqslant 0} E_i \in \mathcal{F}$

The elements of a σ-algebra are called *measurable sets* (or: *events*)

$\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under countable intersections

Probability space

A *probability space* is a structure $(\Omega, \mathcal{F}, \Pr)$ with:

- σ-algebra $(\Omega, \mathcal{F})$
- $\Pr : \mathcal{F} \rightarrow [0, 1]$ is a *probability measure*, i.e.:
  1. $\Pr(\Omega) = 1$, and
  2. $\Pr\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \Pr(E_i)$ for $E_i \in \mathcal{F}$ and $E_i \cap E_j = \varnothing$ for $i \neq j$

$\Pr(E)$ is the probability of E, i.e., E is measurable

Properties of probability measures

- An event E with $\Pr(E) = 1$ is called *almost sure*
  - $\Pr(D) \;=\; \Pr(E \cap D) + \underbrace{\Pr(D \setminus E)}_{=0} \;=\; \Pr(E \cap D)$

- $E_1, \ldots, E_n$ are almost sure implies $\bigcap_{1 \leqslant i \leqslant n} E_i$ is almost sure

- For any Ω and $\mathcal{F} \subseteq 2^{\Omega}$ there exists a *smallest* σ-algebra containing $\mathcal{F}$
  - it is obtained by taking the intersection over all σ-algebras on Ω that contain $\mathcal{F}$
  - this is called the σ-algebra *generated* by $\mathcal{F}$
  - $\mathcal{F}$ is called the *basis* for this σ-algebra

Probability measure on DTMCs

- Events are *infinite paths* in the DTMC $\mathcal{M}$, i.e., $\Omega = \mathit{Paths}(\mathcal{M})$
- σ-algebra on $\mathcal{M}$ is generated by *cylinder sets* of finite paths $\hat{\pi}$:

$$\mathit{Cyl}(\hat{\pi}) \;=\; \left\{ \pi \in \mathit{Paths}(\mathcal{M}) \mid \hat{\pi} \text{ is a prefix of } \pi \right\}$$

  - cylinder sets serve as basis events of the smallest σ-algebra on $\mathit{Paths}(\mathcal{M})$
- $\Pr$ is the *probability measure* on the σ-algebra on $\mathit{Paths}(\mathcal{M})$:

$$\Pr\big(\mathit{Cyl}(s_0 \ldots s_n)\big) \;=\; \iota_{init}(s_0) \cdot \mathbf{P}(s_0 \ldots s_n)$$

  - where $\mathbf{P}(s_0\, s_1 \ldots s_n) \;=\; \prod_{0 \leqslant i < n} \mathbf{P}(s_i, s_{i+1})$
  - and $\mathbf{P}(s_0) = 1$ for paths of length zero

Reachability probabilities

- What is the probability to reach a set of states $B \subseteq S$ in DTMC $\mathcal{M}$?
  - B could be certain *bad* states which should be visited only seldomly
- Which event does $\Diamond B$ mean formally?
  - the union of all cylinders *Cyl*$(s_0 \ldots s_n)$ where
  - $s_0 \ldots s_n$ is an initial path fragment in $\mathcal{M}$ with $s_0, \ldots, s_{n-1} \notin B$ and $s_n \in B$

$$\Pr(\Diamond B) = \sum_{s_0 \ldots s_n \in \mathit{Paths}_{\mathit{fin}}(\mathcal{M}) \cap (S \setminus B)^* B} \Pr\big(\mathit{Cyl}(s_0 \ldots s_n)\big)$$

$$= \sum_{s_0 \ldots s_n \in \mathit{Paths}_{\mathit{fin}}(\mathcal{M}) \cap (S \setminus B)^* B} \iota_{\mathit{init}}(s_0) \cdot \mathbf{P}(s_0 \ldots s_n)$$

Reachability probabilities by infinite sums

Reachability probabilities in finite DTMCs

- Let $\Pr(s \models \Diamond B) = \Pr_s(\Diamond B) = \Pr_s\{\pi \in \mathit{Paths}(s) \mid \pi \models \Diamond B\}$
  - where $\Pr_s$ is the probability measure in $\mathcal{M}$ with only initial state s
- Let variable $x_s = \Pr(s \models \Diamond B)$ for any state s
  - if B is not reachable from s then $x_s = 0$
  - if $s \in B$ then $x_s = 1$
- For any state $s \in \mathit{Pre}^*(B) \setminus B$:

$$x_s = \underbrace{\sum_{t \in S \setminus B} \mathbf{P}(s,t) \cdot x_t}_{\text{reach } B \text{ via } t} + \underbrace{\sum_{u \in B} \mathbf{P}(s,u)}_{\text{reach } B \text{ in one step}}$$

Linear equation system

- These equations can be rewritten into the following form:

$$\mathbf{x} \;=\; \mathbf{A}\mathbf{x} \;+\; \mathbf{b}$$

  - where vector $\mathbf{x} = (x_s)_{s \in \tilde{S}}$ with $\tilde{S} = \mathit{Pre}^*(B) \setminus B$
  - $\mathbf{A} \;=\; \Big(\mathbf{P}(s,t) \Big)_{s,t \in \tilde{S}}$, the transition probabilities in $\tilde{S}$
  - $\mathbf{b} \;=\; \big(b_s \big)_{s \in \tilde{S}}$ contains the probabilities to reach B within one step

- *Linear equation system*: $(\mathbf{I} - \mathbf{A})\mathbf{x} \;=\; \mathbf{b}$
  - note: more than one solution may exist if $\mathbf{I} - \mathbf{A}$ has no inverse (i.e., is singular)

  $\Rightarrow$ characterize the desired probability as least fixed point

Example

Let $B = \{ \textit{delivered} \}$

$\tilde{S} = \{\textit{init}, \textit{try}, \textit{lost}\}$ and the equations:

$$\begin{aligned} x_{\textit{init}} &= x_{\textit{try}} \\ x_{\textit{try}} &= \frac{1}{10} \cdot x_{\textit{lost}} + \frac{9}{10} \\ x_{\textit{lost}} &= x_{\textit{try}} \end{aligned}$$

which can be rewritten as:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{10} \\ 0 & -1 & 1 \end{pmatrix} \cdot \mathbf{x} = \begin{pmatrix} 0 \\ \frac{9}{10} \\ 0 \end{pmatrix}$$

and yields the (unique) solution: $x_{\textit{try}} = x_{\textit{init}} = x_{\textit{lost}} = 1$.

Constrained reachability

- Let $\mathcal{M} = (S, \mathbf{P}, \iota_{init}, AP, L)$ be a (possibly infinite) DTMC and B, $C \subseteq S$

- $C \,\mathsf{U}^{\leqslant n} B$ is the union of the basic cylinders of path fragments:
  - $s_0\, s_1 \ldots s_k$ with $k \leqslant n$ and $s_i \in C$ for all $0 \leqslant i < k$ and $s_k \in B$

- Let $S_{=0}$, $S_{=1}$, $S_?$ be a partition of S such that:
  - $B \subseteq S_{=1} \subseteq \{ s \in S \mid \Pr(s \models C \,\mathsf{U}\, B) = 1 \}$
  - $S \setminus (C \cup B) \subseteq S_{=0} \subseteq \{ s \in S \mid \Pr(s \models C \,\mathsf{U}\, B) = 0 \}$
  - so: all states in $S_?$ belong to $C \setminus B$

- Let $\mathbf{A} = \big(\mathbf{P}(s,t) \big)_{s,t \in S_?}$ and $\big(b_s \big)_{s \in S_?}$ where $b_s = \mathbf{P}(s, S_{=1})$

Least fixed point characterization

The vector $\mathbf{x} = \big(\Pr(s \models C \, \mathsf{U} \, B)\big)_{s \in S_?}$ is the *least fixed point* of the operator

$$\Upsilon : [0,1]^{S_?} \rightarrow [0,1]^{S_?} \quad \text{given by} \quad \Upsilon(\mathbf{y}) = \mathbf{A} \cdot \mathbf{y} + \mathbf{b}$$

Furthermore, for $\mathbf{x}^{(0)} = \mathbf{0}$ and $\mathbf{x}^{(n+1)} = \Upsilon(\mathbf{x}^{(n)})$ for $n \geqslant 0$:

- $\mathbf{x}^{(n)} = (x_s^{(n)})_{s \in S_?}$ where for any s: $x_s^{(n)} = \Pr(s \models C \, \mathsf{U}^{\leqslant n} S_{=1})$
- $\mathbf{x}^{(0)} \leqslant \mathbf{x}^{(1)} \leqslant \mathbf{x}^{(2)} \leqslant \ldots \leqslant \mathbf{x}$, and
- $\mathbf{x} = \lim_{n \rightarrow \infty} \mathbf{x}^{(n)}$

partial ordering is: $\mathbf{y} \leqslant \mathbf{y}'$ iff $y_s \leqslant y'_s$ for all $s \in S_?$

Proof

Expansion law

- Recall in CTL: $\exists(C \, \mathsf{U} \, B)$ is the least solution of expansion law:

$$\exists(C \, \mathsf{U} \, B) \;\equiv\; B \;\vee\; (C \wedge \exists \bigcirc \exists(C \, \mathsf{U} \, B))$$

- That is: the set $X = \mathit{Sat}(\exists(C \, \mathsf{U} \, B))$ is the smallest set such that:

$$B \cup \{\, s \in C \setminus B \mid \mathit{Post}(s) \cap X \neq \varnothing \,\} \;\subseteq\; X$$

- Previous theorem “replaces” $s \in X$ by values x_s in $[0,1]$
  - if $s \in B$ then $x_s = 1$ (compare: $s \in B$ implies $s \in X$)
  - if $s \in S \setminus (C \cup B)$ then $x_s = 0$ (compare: $s \notin C \cup B$ implies $s \notin X$)

- If $s \in C \setminus B$ then $x_s = \sum_{t \in C \setminus B} \mathbf{P}(s,t) \cdot x_t + \sum_{t \in B} \mathbf{P}(s,t)$
  - compare: $s \in C \setminus B$ and $\mathit{Post}(s) \cap X \neq \varnothing$ implies $s \in X$

Constrained reachability probabilities

- So: $\mathbf{x}$ is the *least* solution of $\mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{x}$ in $[0,1]^{S_?}$
- And: can be approximated by:

$$\mathbf{x}^{(0)} = \mathbf{0} \quad \text{and} \quad \mathbf{x}^{(n+1)} = \mathbf{A}\mathbf{x}^{(n)} + \mathbf{b} \ \text{ for } \ n \geqslant 0$$

- *Power method*: compute vectors $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots$ and abort if:

$$\max_{s \in S_?} \left| \, x_s^{(n+1)} - x_s^{(n)} \, \right| < \varepsilon \qquad \text{for some small tolerance } \ \varepsilon$$

  - convergence guaranteed
  - alternative techniques: e.g., Jacobi or Gauss-Seidel, successive overrelaxation

Unique solution

Let $\mathcal{M}$ be a finite DTMC with state space S partitioned into:

- $S_{=0} \; = \; \mathit{Sat}(\neg\exists(C \,\mathsf{U}\, B))$
- $S_{=1}$ a subset of $\{ s \in S \mid \Pr(s \models C \,\mathsf{U}\, B) = 1 \}$ that contains B
- $S_{?} \; = \; S \setminus (S_{=0} \cup S_{=1})$

For B, $C \subseteq S$, the vector

$$\big(\Pr(s \models C \,\mathsf{U}\, B) \big)_{s \in S_?}$$

is the *unique* solution of the linear equation system:

$$\mathbf{x} \; = \; \mathbf{A}\mathbf{x} + \mathbf{b} \quad \text{where} \quad \mathbf{A} \; = \; \big(\mathbf{P}(s,t) \big)_{s,t \in S_?} \;\; \text{and} \;\; \mathbf{b} \; = \; \big(\mathbf{P}(s, S_{=1}) \big)_{s \in S_?}$$

Computing constrained reachability probabilities

- The probabilities of the events $C \,\mathsf{U}^{\leqslant n} B$ can be obtained iteratively:

$$\mathbf{x}^{(0)} = \mathbf{0} \quad \text{and} \quad \mathbf{x}^{(i+1)} = \mathbf{A}\mathbf{x}^{(i)} + \mathbf{b} \text{ for } 0 \leqslant i < n$$

- where $\mathbf{A} = \big(\mathbf{P}(s,t)\big)_{s,t \in C \setminus B}$ and $\mathbf{b} = \big(\mathbf{P}(s,B)\big)_{s \in C \setminus B}$

- Then: $\mathbf{x}^{(n)}(s) = \Pr(s \models C \,\mathsf{U}^{\leqslant n} B)$ for $s \in C \setminus B$

Transient probabilities

- Given that $\mathbf{A}^n(s,t) = \Pr(s \models S_? \, \mathsf{U}^{=n} \, t)$
  - if $B = \varnothing$, $C = S$, we have $S_{=1} = S_{=0} = \varnothing$ and $S_? = S$ and $\mathbf{A} = \mathbf{P}$
  - $\mathbf{P}^n(s,t)$ is the probability to be in state t after n steps once started in s

- Transient probability: $\Theta_n^{\mathcal{M}}(t) = \sum_{s \in S} \iota_{init}(s) \cdot \mathbf{P}^n(s,t)$

- $\Theta_n^{\mathcal{M}} = \underbrace{\mathbf{P} \cdot \mathbf{P} \cdot \ldots \cdot \mathbf{P}}_{n \text{ times}} \cdot \iota_{init} = \mathbf{P}^n \cdot \iota_{init}$
  - where the initial distribution ι_{init} is viewed as column-vector

- Compute $\Theta_n^{\mathcal{M}}$ by successive vector-matrix multiplication:

$$\Theta_0^{\mathcal{M}} = \iota_{init}, \qquad \Theta_n^{\mathcal{M}} = \mathbf{P} \cdot \Theta_{n-1}^{\mathcal{M}} \text{ for } n \geqslant 1$$

Reachability = transient probabilities

- Suppose we want to compute probabilities for $\Diamond^{\leqslant n} B$ in $\mathcal{M}$
  - observe: once B is reached, remaining behaviour is not important
- Adapt $\mathcal{M}$ by making all states in B absorbing
  - $\mathbf{P}_B(s,t) = \mathbf{P}(s,t)$ if $s \notin B$ and $\mathbf{P}_B(s,s) = 1$ for $s \in B$
  - all outgoing transitions of $s \in B$ are replaced by a single self-loop at s
- Then:

$$\underbrace{\Pr^{\mathcal{M}}(\Diamond^{\leqslant n} B)}_{\text{reachability in } \mathcal{M}} = \underbrace{\sum_{s' \in B} \Theta_n^{\mathcal{M}_B}(s')}_{\text{transient probability in } \mathcal{M}_B}$$

Constrained reachability = transient probabilities

- Suppose we want to compute probabilities for $C \, \mathsf{U}^{\leqslant n} B$ in $\mathcal{M}$
  - observe: once B is reached, remaining behaviour is not important
  - observe: once $s \in S \setminus (C \cup B)$ is reached, remaining behaviour not important
- Adapt $\mathcal{M}$ by making all states in B and $S \setminus (C \cup B)$ absorbing
  - $\mathbf{P}_B(s,t) = \mathbf{P}(s,t)$ if $s \notin B$ and $\mathbf{P}_B(s,s) = 1$ for $s \in B$ or $s \in C \cup B$
- Then:

$$\underbrace{\Pr^{\mathcal{M}}(C \, \mathsf{U}^{\leqslant n} B)}_{\text{reachability in } \mathcal{M}} = \underbrace{\sum_{s' \in B} \Theta_n^{\mathcal{M}_{C,B}}(s')}_{\text{transient probability in } \mathcal{M}_{C,B}}$$