Qualitative Properties in Markov Chains

Lecture #20 of Advanced Model Checking

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Discrete-time Markov chains

A DTMC $\mathcal{M}$ is a tuple $(S, \mathbf{P}, \iota_{init}, AP, L)$ with:

- S is a countable nonempty set of states
- $\mathbf{P} : S \times S \rightarrow [0, 1]$, transition probability function s.t. $\sum_{s'} \mathbf{P}(s, s') = 1$
  - $\mathbf{P}(s, s')$ is the probability to jump from s to s' in one step
  - s is absorbing if $\mathbf{P}(s, s) = 1$
- $\iota_{init} : S \rightarrow [0, 1]$, the initial distribution with $\sum\limits_{s \in S} \iota_{init}(s) = 1$
  - $\iota_{init}(s)$ is the probability that system starts in state s
  - state s for which $\iota_{init}(s) > 0$ is an initial state
- $L : S \rightarrow 2^{AP}$, the labelling function

Paths

- State graph of DTMC $\mathcal{M}$
  - vertices are states of $\mathcal{M}$, and $(s, s') \in E$ if and only if $\mathbf{P}(s, s') > 0$
- Paths in $\mathcal{M}$ are maximal (i.e., infinite) paths inits state graph
  - for path π in $\mathcal{M}$, $\mathrm{inf}(\pi)$ is the set of states that are visited infinitely often in π
  - *Paths*$(\mathcal{M})$ and *Paths*$_{fin}(\mathcal{M})$ denote the set of (finite) paths in $\mathcal{M}$
- *Post*$(s) = \{s' \in S \mid \mathbf{P}(s, s') > 0\}$ and *Pre*$(s) = \{s' \in S \mid \mathbf{P}(s', s) > 0\}$
  - *Post*$^*(s)$ is the set of states reachable from s via a finite path fragment
  - *Pre*$^*(s) = \{s' \in S \mid s \in$ *Post*$^*(s')\}$

Probability measure on DTMCs

- Events are *infinite paths* in the DTMC $\mathcal{M}$, i.e., $\Omega = \mathit{Paths}(\mathcal{M})$

- σ-algebra on $\mathcal{M}$ is generated by *cylinder sets* of finite paths $\hat{\pi}$:

$$\mathit{Cyl}(\hat{\pi}) \; = \; \{\, \pi \in \mathit{Paths}(\mathcal{M}) \mid \hat{\pi} \text{ is a prefix of } \pi \,\}$$

  - cylinder sets serve as basis events of the smallest σ-algebra on $\mathit{Paths}(\mathcal{M})$

- $\Pr$ is the *probability measure* on the σ-algebra on $\mathit{Paths}(\mathcal{M})$:

$$\Pr\big(\mathit{Cyl}(s_0 \ldots s_n)\big) \; = \; \iota_{init}(s_0) \cdot \mathbf{P}(s_0 \ldots s_n)$$

  - where $\mathbf{P}(s_0\, s_1 \ldots s_n) \; = \; \prod_{0 \leqslant i < n} \mathbf{P}(s_i, s_{i+1})$
  - and $\mathbf{P}(s_0) = 1$ for paths of length zero

Reachability probabilities

- What is the probability to reach a set of states $B \subseteq S$ in DTMC $\mathcal{M}$?
- Which event does $\Diamond B$ mean formally?
  - the union of all cylinders *Cyl*$(s_0 \ldots s_n)$ where
  - $s_0 \ldots s_n$ is an initial path fragment in $\mathcal{M}$ with $s_0, \ldots, s_{n-1} \notin B$ and $s_n \in B$

$$\begin{aligned}\Pr(\Diamond B) &= \sum_{s_0 \ldots s_n \in \mathit{Paths}_{\mathit{fin}}(\mathcal{M}) \cap (S \setminus B)^* B} \Pr\big(\mathit{Cyl}(s_0 \ldots s_n)\big) \\ &= \sum_{s_0 \ldots s_n \in \mathit{Paths}_{\mathit{fin}}(\mathcal{M}) \cap (S \setminus B)^* B} \iota_{\mathit{init}}(s_0) \cdot \mathbf{P}(s_0 \ldots s_n)\end{aligned}$$

Constrained reachability probabilities

Let $\mathcal{M}$ be a *finite* DTMC with state space S partitioned into:

- $S_{=0} \; = \; \mathit{Sat}(\neg\exists(C \,\mathsf{U}\, B))$
- $S_{=1}$ a subset of $\{ s \in S \mid \Pr(s \models C \,\mathsf{U}\, B) = 1 \}$ that contains B
- $S_{?} \; = \; S \setminus (S_{=0} \cup S_{=1})$

For B, $C \subseteq S$, the vector

$$\big(\Pr(s \models C \,\mathsf{U}\, B) \big)_{s \in S_?}$$

is the *unique* solution of the linear equation system:

$$\mathbf{x} \; = \; \mathbf{A}\mathbf{x} + \mathbf{b} \quad \text{where} \quad \mathbf{A} \; = \; \big(\mathbf{P}(s,t) \big)_{s,t \in S_?} \; \text{ and } \; \mathbf{b} \; = \; \big(\mathbf{P}(s, S_{=1}) \big)_{s \in S_?}$$

Constrained reachability probabilities

- The probabilities of the events $C \,\mathsf{U}^{\leqslant n} B$ can be obtained iteratively:

$$\mathbf{x}^{(0)} = \mathbf{0} \quad \text{and} \quad \mathbf{x}^{(i+1)} = \mathbf{A}\mathbf{x}^{(i)} + \mathbf{b} \text{ for } 0 \leqslant i < n$$

- where $\mathbf{A} = \big(\mathbf{P}(s,t)\big)_{s,t \in C \setminus B}$ and $\mathbf{b} = \big(\mathbf{P}(s,B)\big)_{s \in C \setminus B}$

- Then: $\mathbf{x}^{(n)}(s) = \Pr(s \models C \,\mathsf{U}^{\leqslant n} B)$ for $s \in C \setminus B$

Qualitative properties

- Qualitative analysis of a Markov chain
  - determine whether a certain event holds with probability one or with probability zero
- Randomization versus (strong) fairness
  - visiting a state infinitely often almost surely implies string fairness on its transitions
- Limit behaviour of a finite Markov chain
  - the long run behaviour of a DTMC is captured by reachable strongly connected components
- Almost surely reachability
- Repeated reachability
  - almost sure repeated reachability and probabilities for repeated reachability

Measurability of some events

Let $T \subseteq S$ a subset of states in a (possibly infinite) DTMC.

- The event $\Box\Diamond T$ is measurable
  - $\Box\Diamond T$ can be written as countable intersection of countable unions of cylinder sets:

$$\Box\Diamond T = \bigcap_{n \geqslant 0} \bigcup_{m \geqslant n} \mathit{Cyl}(\text{“}(m{+}1)\text{-st state is in } T\text{”})$$

  - where $\mathit{Cyl}(\ldots)$ is the union of all cylinder sets $\mathit{Cyl}(t_0 \ldots t_m)$ for $t_0 \ldots t_m \in \mathit{Paths}_{\mathit{fin}}(\mathcal{M})$ and $t_m \in T$

- The event $\Box\Diamond\widehat{\pi}$ is measurable
  - then also $\bigcap_{\widehat{\pi} \in \Pi}$ is measurable too

- The event $\Diamond\Box T$ is measurable

Fairness constraints

- *Unconditional fairness*

  no condition is expressed that constrains the circumstances under which something happens

- *Strong fairness*

  if an activity is *infinitely often* enabled (not necessarily always!)
  then it has to be executed infinitely often

- *Weak fairness*

  if an activity is *continuously enabled* (no temporary disabling!)
  then it has to be executed infinitely often

Probabilistic choice as strong fairness

Let $\mathcal{M}$ be a (possibly infinite) Markov chain and s, t states in $\mathcal{M}$.

Then:

$$\Pr(s \models \Box\Diamond t) = \Pr_s\Big(\bigwedge_{\hat{\pi} \in \mathit{Paths}_{fin}(t)} \Box\Diamond\hat{\pi} \Big)$$

where $\bigwedge_{\hat{\pi} \in \mathit{Paths}_{fin}(t)} \Box\Diamond\hat{\pi}$ denotes the set consisting of all paths π such that any path fragment $\hat{\pi} \in \mathit{Paths}_{fin}(t)$ occurs infinitely often in π

Proof

Corollary

For any state s in DTMC $\mathcal{M}$:

$$\Pr(s \models \Box\Diamond t) \;=\; \Pr\Big(s \models \bigwedge_{u \,\in\, \textit{Post}^*(t)} \Box\Diamond u\Big)$$

and

$$\Pr\Big(s \models \bigwedge_{t \in S} \; \bigwedge_{u \in \textit{Post}^*(t)} (\Box\Diamond t \;\longrightarrow\; \Box\Diamond u)\Big) \;=\; 1$$

for finite Markov chains at least one state is visited infinitely often on all paths

Graph notions

Let $\mathcal{M} = (S, \mathbf{P}, \iota_{init}, AP, L)$ be a *finite* Markov chain

- $T \subseteq S$ is *strongly connected* if:
  - $s \in T$ and $t \in T$ are mutually reachable via edges in T
- T is a *strongly connected component* (SCC) of $\mathcal{M}$ if:
  - it is strongly connected and no proper superset of T is strongly connected
- T is a *bottom SCC* (BSCC) if:
  - it is an SCC and no state outside T is reachable from T
  - for any state $t \in T$ it holds $\mathbf{P}(s, T) = \sum_{t \in T} \mathbf{P}(s, t) = 1$
  - let *BSCC*$(\mathcal{M})$ denote the set of BSCCs of $\mathcal{M}$

Limit behaviour

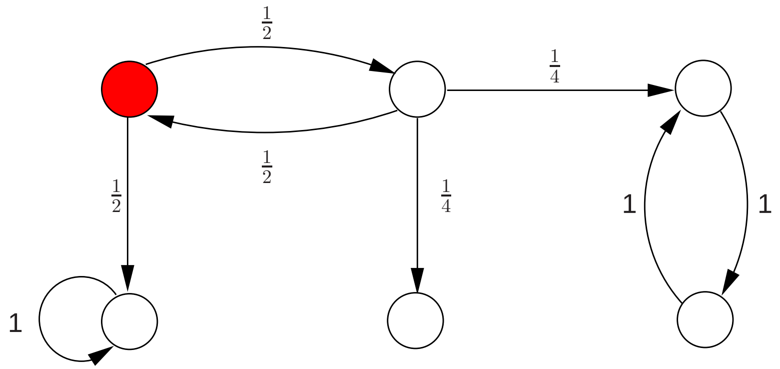
For each state s of a finite Markov chain $\mathcal{M}$:

$$\Pr_s\{\, \pi \in \textit{Paths}(s) \mid \inf(\pi) \in \textit{BSCC}(\mathcal{M}) \,\} = 1$$

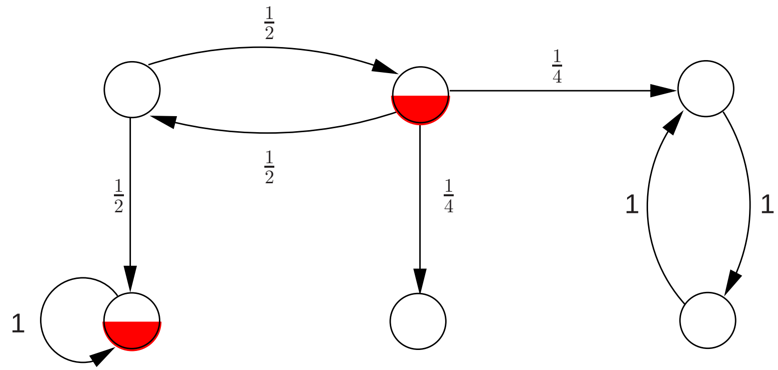
almost surely any finite DTMC eventually reaches a BSCC
and visits all its states ∞ often

Proof

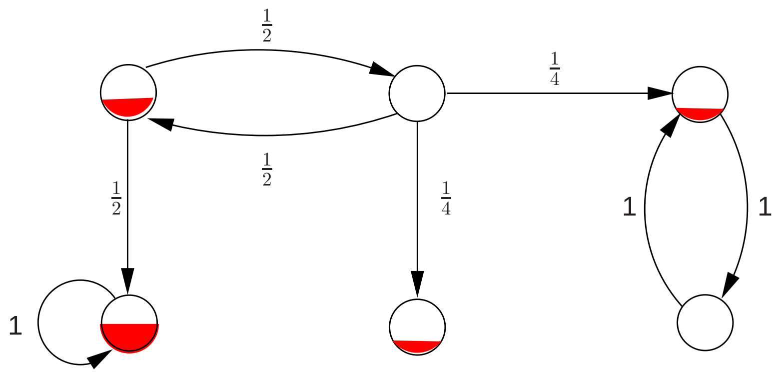
Evolution of an example DTMC



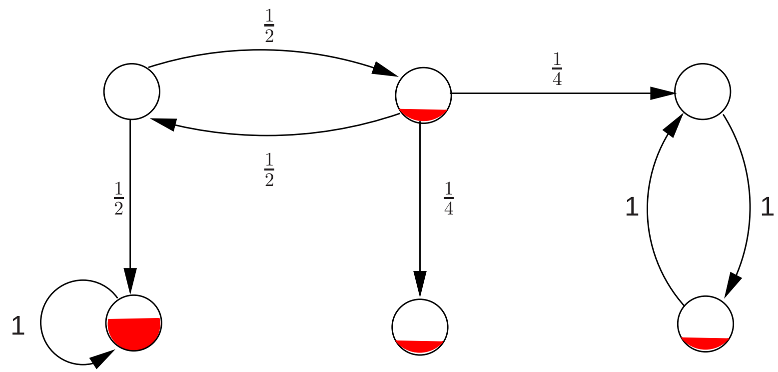
zero-th epoch



first epoch

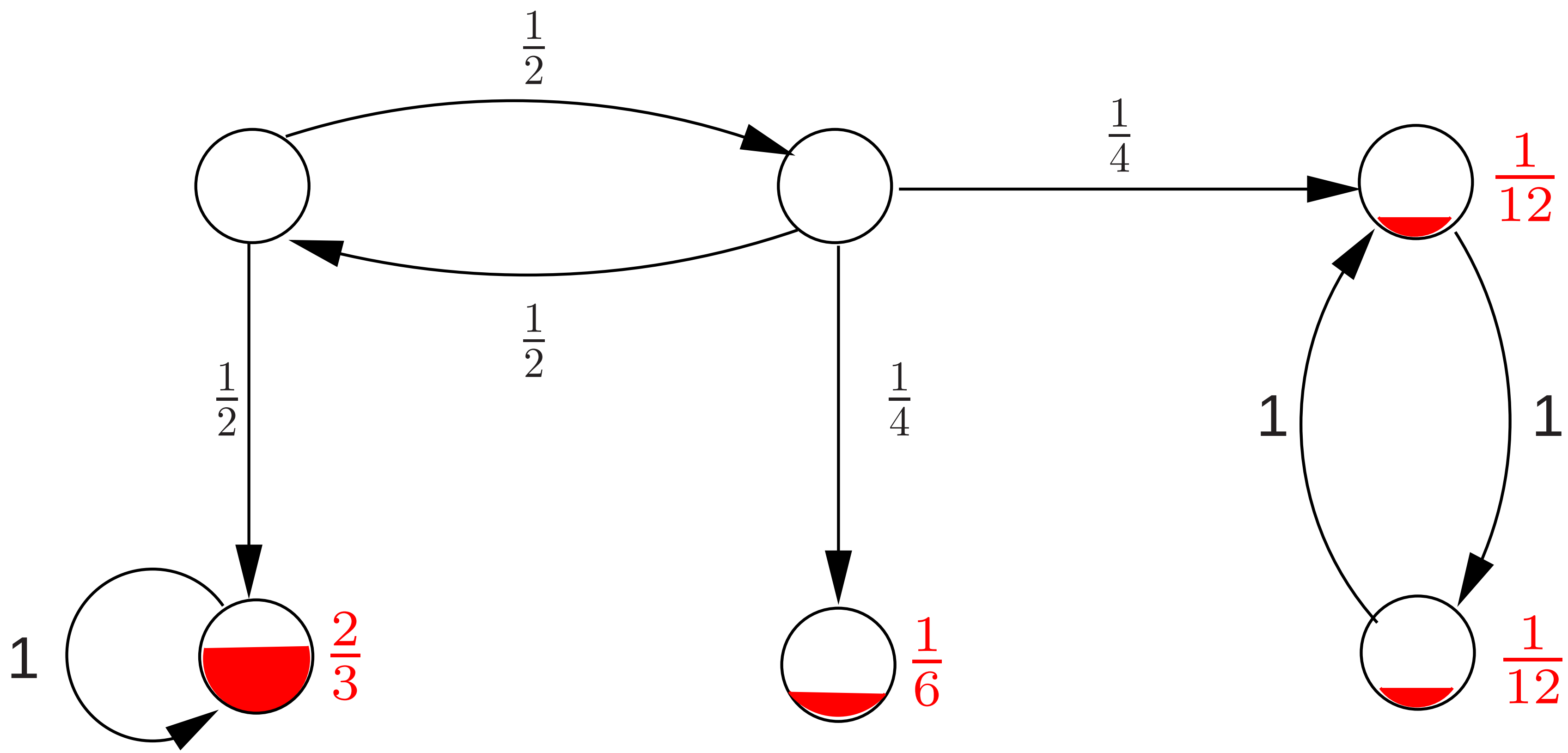


second epoch



third epoch

On the long run



Almost sure reachability

Let $\mathcal{M}$ be a finite DTMC with state space S, $s \in S$.

For $B \subseteq S$ a set of absorbing states the following statements are equivalent:

(a) $\Pr(s \models \Diamond B) = 1$

(b) $\mathit{Post}^*(t) \cap B \neq \varnothing$ for any state $t \in \mathit{Post}^*(s)$

(c) $s \in S \setminus \mathit{Pre}^*\big(S \setminus \mathit{Pre}^*(B)\big)$

in particular, $\{ s \in S \mid \Pr(s \models \Diamond B) = 1 \} = S \setminus \mathit{Pre}^*\Big(S \setminus \mathit{Pre}^*(B)\Big)$

Proof

Almost sure reachability

- Given finite DTMC $\mathcal{M}$ and set of states $B \subseteq S$, determine:

$$s \in S \quad \text{such that} \quad \Pr(s \models \Diamond B) = 1$$

1. Make all states in B absorbing (yielding $\mathcal{M}_B$)
2. Determine $S \setminus \mathit{Pre}^*\big(\, S \setminus \mathit{Pre}^*(B)\,\big)$ by a graph analysis
   - do a backward search from B in $\mathcal{M}_B$ to determine $\mathit{Pre}^*(B)$
   - then a backward search from $S \setminus \mathit{Pre}^*(B)$ in $\mathcal{M}_B$

- Time complexity: linear in the size of $\mathcal{M}$

0-1 constrained reachability

For finite DTMC $\mathcal{M}$, the sets:

$$S_{=0} = \{ s \in S \mid \Pr(s \models C \,\mathsf{U}\, B) = 0 \} \quad \text{and}$$

$$S_{=1} = \{ s \in S \mid \Pr(s \models C \,\mathsf{U}\, B) = 1 \}$$

can be computed in time $\mathcal{O}(\mathit{size}(\mathcal{M}))$

Proof

Repeated reachability (1)

Let $\mathcal{M}$ be a finite DTMC with state-space S.

For $B \subseteq S$, $s \in S$, the following statements are equivalent:

(a) $\Pr(s \models \Box\Diamond B) = 1$

(b) $T \cap B \neq \varnothing$ for each BSCC T reachable from s

(c) $s \models \forall\Box\exists\Diamond B$

$\{ s \in S \mid \Pr(s \models \Box\Diamond B) = 1 \}$ *can be determined in* $\mathcal{O}($*size*$(\mathcal{M}))$

Repeated reachability (2)

Let $\mathcal{M}$ be a finite DTMC with state space S and $B \subseteq S$, $s \in S$

Let U be the union of all BSCCs T with $T \cap B \neq \varnothing$

Then:

$$\Pr(s \models \Box\Diamond B) \; = \; \Pr(s \models \Diamond U)$$

Persistence properties

- $\Pr(s \models \Diamond\Box B) = 1$ iff $T \subseteq B$ for any BSCC T reachable from s
- $\Pr(s \models \Diamond\Box B) = \Pr(s \models \Diamond V)$
  - where V is the union of all BSCCs T with $T \subseteq B$

Summary

Any considered qualitative property on finite DTMCs can be checked

by a graph analysis of the underlying state graph

Infinite DTMCs