Foundations of Informatics: a Bridging Course

Week 3: Formal Languages and Semantics
Part B: Context-Free Languages

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http://cosec.bit.uni-bonn.de/students/teaching/11us/11us-bridgingcourse/
http://www-i2.informatik.rwth-aachen.de/i2/b-it11/

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Outline of Part B

Introductory Example I

Example B.1

Syntax definition of programming languages by “Backus-Naur” rules

Here: simple arithmetic expressions

$$
\begin{aligned}
\langle \mathit{Expression} \rangle \quad ::= \quad & 0 \\
\mid \quad & 1 \\
\mid \quad & \langle \mathit{Expression} \rangle + \langle \mathit{Expression} \rangle \\
\mid \quad & \langle \mathit{Expression} \rangle * \langle \mathit{Expression} \rangle \\
\mid \quad & (\langle \mathit{Expression} \rangle)
\end{aligned}
$$

Meaning:

> *An expression is either 0 or 1, or it is of the form* $u + v$, $u * v$, *or* (u) *where* u, v *are again expressions*

Introductory Example II

Example B.2 (continued)

Here we abbreviate $\langle \mathit{Expression} \rangle$ as E, and use "$\rightarrow$" instead of "$::=$".

Thus:

$$E \rightarrow 0 \mid 1 \mid E + E \mid E * E \mid (E)$$

Introductory Example II

Example B.2 (continued)

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Thus:

$$E \rightarrow 0 \mid 1 \mid E + E \mid E * E \mid (E)$$

Now expressions can be generated by applying rules to the start symbol E:

$$\begin{aligned} E &\Rightarrow E * E \\ &\Rightarrow (E) * E \\ &\Rightarrow (E) * 1 \\ &\Rightarrow (E + E) * 1 \\ &\Rightarrow (0 + E) * 1 \\ &\Rightarrow (0 + 1) * 1 \end{aligned}$$

Context-Free Grammars I

Definition B.3

A context-free grammar (CFG) is a quadruple

$$G = \langle N, \Sigma, P, S \rangle$$

where

- N is a finite set of nonterminal symbols
- Σ is the (finite) alphabet of terminal symbols (disjoint from N)
- P is a finite set of production rules of the form $A \to \alpha$ where $A \in N$ and $\alpha \in (N \cup \Sigma)^*$
- $S \in N$ is a start symbol

Context-Free Grammars II

Example B.4

For the above example, we have:

- $N = \{E\}$
- $\Sigma = \{0, 1, +, *, (,)\}$
- $P = \{E \to 0, E \to 1, E \to E + E, E \to E * E, E \to (E)\}$
- $S = E$

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- $S = E$

Naming conventions:

- nonterminals start with uppercase letters
- terminals start with lowercase letters
- start symbol = symbol on LHS of first production

$\Longrightarrow$ grammar completely defined by productions

Definition B.5

Let $G = \langle N, \Sigma, P, S \rangle$ be a CFG.

- A sentence $\gamma \in (N \cup \Sigma)^*$ is directly derivable from $\beta \in (N \cup \Sigma)^*$ if there exist $\pi = A \to \alpha \in P$ and $\delta_1, \delta_2 \in (N \cup \Sigma)^*$ such that $\beta = \delta_1 A \delta_2$ and $\gamma = \delta_1 \alpha \delta_2$ (notation: $\beta \overset{\pi}{\Rightarrow} \gamma$ or just $\beta \Rightarrow \gamma$) .
- A derivation (of length n) of γ from β is a sequence of direct derivations of the form $\delta_0 \Rightarrow \delta_1 \Rightarrow \ldots \Rightarrow \delta_n$ where $\delta_0 = \beta$, $\delta_n = \gamma$, and $\delta_{i-1} \Rightarrow \delta_i$ for every $1 \leq i \leq n$ (notation: $\beta \Rightarrow^* \gamma$).
- A word $w \in \Sigma^*$ is called derivable in G if $S \Rightarrow^* w$.

Context-Free Languages I

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Let $G = \langle N, \Sigma, P, S \rangle$ be a CFG.

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- A derivation (of length n) of γ from β is a sequence of direct derivations of the form $\delta_0 \Rightarrow \delta_1 \Rightarrow \ldots \Rightarrow \delta_n$ where $\delta_0 = \beta$, $\delta_n = \gamma$, and $\delta_{i-1} \Rightarrow \delta_i$ for every $1 \leq i \leq n$ (notation: $\beta \Rightarrow^* \gamma$).
- A word $w \in \Sigma^*$ is called derivable in G if $S \Rightarrow^* w$.
- The language generated by G is $L(G) := \{w \in \Sigma^* \mid S \Rightarrow^* w\}$.
- A language $L \subseteq \Sigma^*$ is called context-free (CFL) if it is generated by some CFG.
- Two grammars G_1, G_2 are equivalent if $L(G_1) = L(G_2)$.

Context-Free Languages II

Example B.6

The language $\{a^n b^n \mid n \geq 1\}$ is context-free (but not regular—see Ex. A.59). It is generated by the grammar $G = \langle N, \Sigma, P, S \rangle$ with

- $N = \{S\}$
- $\Sigma = \{a, b\}$
- $P = \{S \to aSb \mid ab\}$

(proof: on the board)

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- $\Sigma = \{a, b\}$
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(proof: on the board)

Remark: illustration of derivations by derivation trees

- root labeled by start symbol
- leafs labeled by terminal symbols
- successors of node labeled according to right-hand side of production rule

(example on the board)

Context-Free Grammars and Languages

Seen:

- Context-free grammars
- Derivations
- Context-free languages

Context-Free Grammars and Languages

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- Context-free grammars
- Derivations
- Context-free languages

Open:

- Relation between context-free and regular languages

Outline of Part B

Context-Free and Regular Languages

Theorem B.7

1. *Every regular language is context-free.*
2. *There exist CFLs which are not regular.*

(In other words: the class of regular languages is a proper subset of the class of CFLs.)

Context-Free and Regular Languages

Theorem B.7

1. *Every regular language is context-free.*
2. *There exist CFLs which are not regular.*

(In other words: the class of regular languages is a proper subset of the class of CFLs.)

Proof.

1. Let L be a regular language, and let $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA which recognizes L. $G := \langle N, \Sigma, P, S \rangle$ is defined as follows:
   - $N := Q$, $S := q_0$
   - if $\delta(q, a) = q'$, then $q \to aq' \in P$
   - if $q \in F$, then $q \to \varepsilon \in P$

   Obviously a w-labeled run in $\mathfrak{A}$ from q_0 to F corresponds to a derivation of w in G, and vice versa. Thus $L(\mathfrak{A}) = L(G)$ (example on the board).
2. A counterexample is $\{a^n b^n \mid n \geq 1\}$ (see Ex. A.59 and B.6).

Seen:

- CFLs are more expressive than regular languages

Context-Free Grammars and Languages

Seen:

- CFLs are more expressive than regular languages

Open:

- Decidability of word problem

Outline of Part B

The Word Problem

- **Goal:** given $G = \langle N, \Sigma, P, S \rangle$ and $w \in \Sigma^*$, decide whether $w \in L(G)$ or not
- For regular languages this was easy: just let the corresponding DFA run on w.
- But here: how to decide when to stop a derivation?
- **Solution:** establish normal form for grammars which guarantees that each nonterminal produces at least one terminal symbol

$\Longrightarrow$ only finitely many combinations to be inspected

Chomsky Normal Form I

Definition B.8

A CFG is in Chomsky Normal Form (Chomsky NF) if every of its productions is of the form

$$A \to BC \quad \text{or} \quad A \to a$$

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Example B.9

Let $S \to ab \mid aSb$ be the grammar which generates $L := \{a^n b^n \mid n \geq 1\}$.
An equivalent grammar in Chomsky NF is

$$\begin{aligned} S &\to AB \mid AC && \text{(generates } L\text{)} \\ A &\to a && \text{(generates } \{a\}\text{)} \\ B &\to b && \text{(generates } \{b\}\text{)} \\ C &\to SB && \text{(generates } \{a^n b^{n+1} \mid n \geq 1\}\text{)} \end{aligned}$$

Chomsky Normal Form II

Theorem B.10

Every CFL L with $\varepsilon \notin L$ is generatable by a CFG in Chomsky NF.

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Every CFL L with $\varepsilon \notin L$ is generatable by a CFG in Chomsky NF.

Proof.

Let L be a CFL, and let $G = \langle N, \Sigma, P, S \rangle$ be some CFG which generates L. The transformation of P into rules of the form $A \to BC$ and $A \to a$ proceeds in three steps:

1. terminal symbols only in rules of the form $A \to a$ (thus all other rules have the shape $A \to A_1 \dots A_n$)
2. elimination of "chain rules" of the form $A \to B$
3. elimination of rules of the form $A \to A_1 \dots A_n$ where $n > 2$

□

Chomsky Normal Form III

Proof of Theorem B.10 (continued).

Step 1: (only $A \to a$)

1. let $N' := \{B_a \mid a \in \Sigma\}$
2. let $P' := \{A \to \alpha' \mid A \to \alpha \in P\} \cup \{B_a \to a \mid a \in \Sigma\}$
   where α' is obtained from α by replacing every $a \in \Sigma$ with B_a

This yields G' (example: on the board)

Proof of Theorem B.10 (continued).

Step 1: (only $A \to a$)

1. let $N' := \{B_a \mid a \in \Sigma\}$
2. let $P' := \{A \to \alpha' \mid A \to \alpha \in P\} \cup \{B_a \to a \mid a \in \Sigma\}$ where α' is obtained from α by replacing every $a \in \Sigma$ with B_a

This yields G' (example: on the board)

Step 2: (elimination of $A \to B$)

1. determine all derivations $A_1 \Rightarrow \ldots \Rightarrow A_n$ with rules of the form $A \to B$ without repetition of nonterminals ($\Longrightarrow$ only finitely many!)
2. let $$P'' := (P \cup \{A_1 \to \alpha \mid A_1 \Rightarrow \ldots \Rightarrow A_n \Rightarrow \alpha, \alpha \notin N\}) \setminus \{A \to B \mid A \to B \in P'\}$$

This yields G'' (example: on the board)

Chomsky Normal Form IV

Proof of Theorem B.10 (continued).

Step 3: for every $A \to A_1 \dots A_n$ with $n > 2$:

1. add new symbols $B_1, \dots, B_{n-2}$ to N''
2. replace $A \to A_1 \dots A_n$ by

$$\begin{array}{rcl} A & \to & A_1 B_1 \\ B_1 & \to & A_2 B_2 \\ & \vdots & \\ B_{n-3} & \to & A_{n-2} B_{n-2} \\ B_{n-2} & \to & A_{n-1} A_n \end{array}$$

This yields G''' (example: on the board)

One can show: G, G', G'', G''' are equivalent □

The Word Problem Revisited

Goal: given $w \in \Sigma^+$ and $G = \langle N, \Sigma, P, S \rangle$ such that $\varepsilon \notin L(G)$, decide if $w \in L(G)$ or not

(If $w = \varepsilon$, then $w \in L(G)$ easily decidable for arbitrary G)

Approach by Cocke, Younger, Kasami (CYK algorithm):

1. transform G into Chomsky NF
2. let $w = a_1 \dots a_n$ $(n \geq 1)$
3. let $w[i,j] := a_i \dots a_j$ for every $1 \leq i \leq j \leq n$
4. consider segments $w[i,j]$ in order of increasing length, starting with $w[i,i]$ (i.e., single letters)
5. in each case, determine $N_{i,j} := \{A \in N \mid A \Rightarrow^* w[i,j]\}$
6. test whether $S \in N_{1,n}$ (and thus, whether $S \Rightarrow^* w[1,n] = w$)

The CYK Algorithm I

Algorithm B.11 (CYK Algorithm)

Input: $G = \langle N, \Sigma, P, S \rangle$ *in Chomsky NF,* $w = a_1 \ldots a_n \in \Sigma^+$

Question: $w \in L(G)$*?*

Procedure:

```
for i := 1 to n do
   N_{i,i} := {A ∈ N | A → a_i ∈ P}
next i
for d := 1 to n − 1 do    % compute N_{i,i+d}
   for i := 1 to n − d do
      j := i + d; N_{i,j} := ∅;
      for k := i to j − 1 do
         N_{i,j} := N_{i,j} ∪ {A ∈ N | there is A → BC ∈ P
                                 with B ∈ N_{i,k}, C ∈ N_{k+1,j}}
      next k
   next i
next d
```

Output: *"yes" if* $S \in N_{1,n}$*, otherwise "no"*

The CYK Algorithm II

Example B.12

- $G: \quad S \to SA \mid a$
  $\quad\quad A \to BS$
  $\quad\quad B \to BB \mid BS \mid b \mid c$
- $w = abaaba$
- Matrix representation of $N_{i,j}$

(on the board)

The Word Problem for Context-Free Languages

Seen:

- Word problem decidable using CYK algorithm

The Word Problem for Context-Free Languages

Seen:

- Word problem decidable using CYK algorithm

Open:

- Emptiness problem

Outline of Part B

The Emptiness Problem

- **Goal:** given $G = \langle N, \Sigma, P, S \rangle$, decide whether $L(G) = \emptyset$ or not
- For regular languages this was easy: check in the corresponding DFA whether some final state is reachable from the initial state.
- Here: test whether start symbol is productive, i.e., whether it generates a terminal word

The Productivity Test

Algorithm B.13 (Productivity Test)

Input: $G = \langle N, \Sigma, P, S \rangle$

Question: $L(G) = \emptyset$?

Procedure: *mark every* $a \in \Sigma$ *as productive*;

```
repeat
   if there is A → α ∈ P such that
        all symbols in α productive then
      mark A as productive;
   end;
until no further productive symbols found;
```

Output: *"no" if* S *productive, otherwise "yes"*

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Output: *"no" if* S *productive, otherwise "yes"*

Example B.14

$$G: \quad \begin{aligned} S &\to AB \mid CA \\ A &\to a \\ B &\to BC \mid AB \\ C &\to aB \mid b \end{aligned}$$

(on the board)

The Emptiness Problem for CFLs

Seen:

- Emptiness problem decidable using productivity test

The Emptiness Problem for CFLs

Seen:

- Emptiness problem decidable using productivity test

Open:

- Characterizing automata model

Outline of Part B

Pushdown Automata I

- **Goal:** introduce an automata model which exactly accepts CFLs
- **Clear:** DFA not sufficient
  (missing "counting capability", e.g. for $\{a^n b^n \mid n \geq 1\}$)
- DFA will be extended to pushdown automata by
  - adding a pushdown store which stores symbols from a pushdown alphabet and uses a specific bottom symbol
  - adding push and pop operations to transitions

Pushdown Automata II

Definition B.15

A pushdown automaton (PDA) is of the form $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ where

- Q is a finite set of states
- Σ is the (finite) input alphabet
- Γ is the (finite) pushdown alphabet
- $\Delta \subseteq (Q \times \Gamma \times \Sigma_\varepsilon) \times (Q \times \Gamma^*)$ is a finite set of transitions
- $q_0 \in Q$ is the initial state
- Z_0 is the (pushdown) bottom symbol
- $F \subseteq Q$ is a set of final states

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- $q_0 \in Q$ is the initial state
- Z_0 is the (pushdown) bottom symbol
- $F \subseteq Q$ is a set of final states

Interpretation of $((q, Z, x), (q', \delta)) \in \Delta$: if the PDA $\mathfrak{A}$ is in state q where Z is on top of the stack and x is the next input symbol (or empty), then $\mathfrak{A}$ reads x, replaces Z by δ, and changes into the state q'.

Configurations, Runs, Acceptance

Definition B.16

Let $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ be a PDA.

- An element of $Q \times \Gamma^* \times \Sigma^*$ is called a configuration of $\mathfrak{A}$.
- The initial configuration for input $w \in \Sigma^*$ is given by (q_0, Z_0, w).
- The set of final configurations is given by $F \times \{\varepsilon\} \times \{\varepsilon\}$.
- If $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$ for every $\gamma \in \Gamma^*$, $w \in \Sigma^*$.
- $\mathfrak{A}$ accepts $w \in \Sigma^*$ if $(q_0, Z_0, w) \vdash^* (q, \varepsilon, \varepsilon)$ for some $q \in F$.
- The language accepted by $\mathfrak{A}$ is $L(\mathfrak{A}) := \{w \in \Sigma^* \mid \mathfrak{A} \text{ accepts } w\}$.
- A language L is called PDA-recognizable if $L = L(\mathfrak{A})$ for some PDA $\mathfrak{A}$.
- Two PDA $\mathfrak{A}_1, \mathfrak{A}_2$ are called equivalent if $L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$.

Examples

Example B.17

1. PDA which recognizes $L = \{a^n b^n \mid n \geq 1\}$
(on the board)

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2. PDA which recognizes $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length; on the board)

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1. PDA which recognizes $L = \{a^n b^n \mid n \geq 1\}$
   (on the board)
2. PDA which recognizes $L = \{ww^R \mid w \in \{a, b\}^*\}$
   (palindromes of even length; on the board)

Observation: $\mathfrak{A}_2$ is nondeterministic: whenever a construction transition is applicable, the pushdown could also be deconstructed

Deterministic PDA

Definition B.18

A PDA $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is called deterministic (DPDA) if for every $q \in Q, Z \in \Gamma$,

1. for every $x \in \Sigma_\varepsilon$, at most one (q, Z, x)-transition in Δ and
2. if there is a (q, Z, a)-transition in Δ for some $a \in \Sigma$, then there is no (q, Z, ε)-transition in Δ.

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2. if there is a (q, Z, a)-transition in Δ for some $a \in \Sigma$, then there is no (q, Z, ε)-transition in Δ.

Remark: this excludes two types of nondeterminism:

1. if $((q, Z, x), (q'_1, \delta_1)), ((q, Z, x), (q'_2, \delta_2)) \in \Delta$:
$$(q'_1, \delta_1\gamma, w) \dashv (q, Z\gamma, xw) \vdash (q'_2, \delta_2\gamma, w)$$
2. if $((q, Z, a), (q'_1, \delta_1)), ((q, Z, \varepsilon), (q'_2, \delta_2)) \in \Delta$:
$$(q'_1, \delta_1\gamma, w) \dashv (q, Z\gamma, aw) \vdash (q'_2, \delta_2\gamma, aw)$$

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2. if $((q, Z, a), (q_1', \delta_1)), ((q, Z, \varepsilon), (q_2', \delta_2)) \in \Delta$:
$$(q_1', \delta_1\gamma, w) \dashv (q, Z\gamma, aw) \vdash (q_2', \delta_2\gamma, aw)$$

Corollary B.19

In a DPDA, every configuration has at most one $\vdash$-successor.

Expressiveness of DPDA

One can show: determinism restricts the set of acceptable languages (DPDA-recognizable languages are closed under complement, which is generally not true for PDA-recognizable languages)

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Example B.20

The set of palindromes of even length is PDA-recognizable, but not DPDA-recognizable (without proof).

PDA and Context-Free Languages I

Theorem B.21

A language is context-free iff it is PDA-recognizable.

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Proof.

$\Longleftarrow$ omitted

$\Longrightarrow$ let $G = \langle N, \Sigma, P, S \rangle$ be a CFG. Construction of PDA $\mathfrak{A}_G$ recognizing $L(G)$:

- $\mathfrak{A}_G$ simulates a derivation of G where the leftmost nonterminal of a sentence form is replaced ("leftmost derivation")
- begin with S on pushdown
- if nonterminal on top: apply a corresponding production rule
- if terminal on top: match with next input symbol

□

PDA and Context-Free Languages II

Proof of Theorem B.21 (continued).

$\Longrightarrow$ Formally: $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- if $A \to \alpha \in P$, then $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$
- if $a \in \Sigma$, then $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$
- $Z_0 := S$
- $F := Q$

$\square$

PDA and Context-Free Languages II

Proof of Theorem B.21 (continued).

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- $Z_0 := S$
- $F := Q$

□

Example B.22

"Bracket language", given by G:

$$S \to \langle\rangle \mid \langle S \rangle \mid SS$$

(on the board)

Seen:

- Definition of PDA
- Equivalence of PDA-recognizable and context-free languages

Pushdown Automata

Seen:

- Definition of PDA
- Equivalence of PDA-recognizable and context-free languages

Open:

- Closure properties of CFLs

Outline of Part B

Positive Results

Theorem B.23

The set of CFLs is closed under concatenation, union, and iteration.

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Proof.

For $i = 1, 2$, let $G_i = \langle N_i, \Sigma, P_i, S_i \rangle$ with $L_i := L(G_i)$ and $N_1 \cap N_2 = \emptyset$.
Then

Positive Results

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- $G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1 \cup N_2$ and $P := \{S \to S_1 S_2\} \cup P_1 \cup P_2$ generates $L_1 \cdot L_2$;

Positive Results

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For $i = 1, 2$, let $G_i = \langle N_i, \Sigma, P_i, S_i \rangle$ with $L_i := L(G_i)$ and $N_1 \cap N_2 = \emptyset$. Then

- $G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1 \cup N_2$ and $P := \{S \to S_1 S_2\} \cup P_1 \cup P_2$ generates $L_1 \cdot L_2$;
- $G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1 \cup N_2$ and $P := \{S \to S_1 \mid S_2\} \cup P_1 \cup P_2$ generates $L_1 \cup L_2$; and

Positive Results

Theorem B.23

The set of CFLs is closed under concatenation, union, and iteration.

Proof.

For $i = 1, 2$, let $G_i = \langle N_i, \Sigma, P_i, S_i \rangle$ with $L_i := L(G_i)$ and $N_1 \cap N_2 = \emptyset$. Then

- $G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1 \cup N_2$ and $P := \{S \to S_1 S_2\} \cup P_1 \cup P_2$ generates $L_1 \cdot L_2$;
- $G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1 \cup N_2$ and $P := \{S \to S_1 \mid S_2\} \cup P_1 \cup P_2$ generates $L_1 \cup L_2$; and
- $G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1$ and $P := \{S \to \varepsilon \mid S_1 S\} \cup P_1$ generates L_1^*.

□

Negative Results

Theorem B.24

The set of CFLs is not closed under intersection and complement.

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Proof.

- Both $L_1 := \{a^k b^k c^l \mid k, l \in \mathbb{N}\}$ and $L_2 := \{a^k b^l c^l \mid k, l \in \mathbb{N}\}$ are CFLs, but not $L_1 \cap L_2 = \{a^n b^n c^n \mid n \in \mathbb{N}\}$ (without proof).

Negative Results

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- If CFLs were closed under complement, then also under intersection (as $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$).

□

Overview of Decidability and Closure Results

Decidability Results			
	$w \in L$	$L = \emptyset$	$L_1 = L_2$
Reg	+ (A.38)	+ (A.40)	+ (A.42)
CFL	+ (B.11)	+ (B.13)	–

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Decidability Results			
	$w \in L$	$L = \emptyset$	$L_1 = L_2$
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CFL	+ (B.11)	+ (B.13)	–

Closure Results					
	$L_1 \cdot L_2$	$L_1 \cup L_2$	$L_1 \cap L_2$	$\overline{L}$	L^*
Reg	+ (A.28)	+ (A.18)	+ (A.16)	+ (A.14)	+ (A.29)
CFL	+ (B.23)	+ (B.23)	– (B.24)	– (B.24)	+ (B.23)

Outline of Part B

Outlook

- Equivalence problem for CFG and PDA ("$L(X_1) = L(X_2)$?") (generally undecidable, decidable for DPDA)
- Pumping Lemma for CFL
- Greibach Normal Form for CFG
- Construction of parsers for compilers
- Non-context-free grammars and languages (context-sensitive and recursively enumerable languages, Turing machines—see Week 4)