Foundations of Informatics: a Bridging Course
Week 3: Formal Languages and Semantics
Part A: Regular Languages

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b-it, Bonn, Winter Semester 2013/14

Organization

- Schedule:
    - lecture 9:00-10:30, 11:00-12:30 (Mon-Fri)
        - 9:30-11:00, 11:15-12:45?
    - exercises 14:00-14:45, 15:15-16:00 (Mon-Thu)
        - 14:00-15:30?
- Exam on Monday, 10 March 2014, 14:00, b-it
- Please ask questions!

Overview of Week 3

1. Regular Languages
2. Context-Free Languages
3. Processes and Concurrency

Literature

- J.E. Hopcroft, R. Motwani, J.D. Ullmann: *Introduction to Automata Theory, Languages, and Computation*, 2nd ed., Addison-Wesley, 2001
- A. Asteroth, C. Baier: *Theoretische Informatik*, Pearson Studium, 2002 [in German]
- http://www.jflap.org/
  (software for experimenting with formal languages and automata)

Outline of Part A

Words and Languages

- Computer systems transform data
- Data encoded as (binary) words

$\implies$ Data sets = sets of words = formal languages,
data transformations = functions on words

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$\implies$ Data sets = sets of words = formal languages, data transformations = functions on words

---

Example A.1

$Java = \{$all valid Java programs$\}$,

$Compiler : Java \rightarrow Bytecode$

Alphabets

The atomic elements of words are called symbols (or letters).

Definition A.2

An alphabet is a finite, non-empty set of symbols ("letters").

$\Sigma, \Gamma, \ldots$ denote alphabets

$a, b, \ldots$ denote letters

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2. Latin alphabet $\Sigma_{\text{latin}} := \{a, b, c, \ldots, z\}$

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3. Keyboard alphabet Σ_{key}
4. Morse alphabet $\Sigma_{\text{morse}} := \{\cdot, -, \sqcup\}$

Words

Definition A.4

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- The **concatenation** of two words $v = a_1 \ldots a_m$ ($m \in \mathbb{N}$) and $w = b_1 \ldots b_n$ ($n \in \mathbb{N}$) is the word

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- A **prefix/suffix** v of a word w is an initial/trailing part of w, i.e., $w = vv'/w = v'v$ for some $v' \in \Sigma^*$.
- If $w = a_1 \ldots a_n$, then $w^R := a_n \ldots a_1$.

Formal Languages I

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1. over $\mathbb{B} = \{0, 1\}$: set of all bit strings containing 1101
2. over $\Sigma = \{I, V, X, L, C, D, M\}$: set of all valid roman numbers
3. over Σ_{key}: set of all valid Java programs

Formal Languages II

Seen:

- Basic notions: alphabets, words
- Formal languages as sets of words

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- Basic notions: alphabets, words
- Formal languages as sets of words

Open:

- Description of computations on words?

Outline of Part A

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Example: Pattern Matching

Example A.7 (Pattern 1101)

1. Read Boolean string bit-by-bit
2. Test whether it contains 1101
3. Idea: remember which (initial) part of 1101 has been recognized
4. Five prefixes: ε, 1, 11, 110, 1101
5. Diagram: on the board

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What we used:

- finitely many (storage) states
- an initial state
- for every current state and every input symbol: a new state
- a successful state

Deterministic Finite Automata I

Definition A.8

A deterministic finite automaton (DFA) is of the form

$$\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$$

where

- Q is a finite set of states
- Σ denotes the input alphabet
- $\delta : Q \times \Sigma \to Q$ is the transition function
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of final (or: accepting) states

Deterministic Finite Automata II

Example A.9

Pattern matching (Example A.7):

- $Q = \{q_0, \ldots, q_4\}$
- $\Sigma = \mathbb{B} = \{0, 1\}$
- $\delta : Q \times \Sigma \to Q$ on the board
- $F = \{q_4\}$

Deterministic Finite Automata II

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- $F = \{q_4\}$

Graphical Representation of DFA:

- states $\implies$ nodes
- $\delta(q, a) = q' \implies q \xrightarrow{a} q'$
- initial state: incoming edge without source state
- final state(s): double circle

Acceptance by DFA I

Definition A.10

Let $\langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA. The extension of $\delta : Q \times \Sigma \to Q$,

$$\delta^* : Q \times \Sigma^* \to Q,$$

is defined by

$$\delta^*(q, w) := \text{state after reading } w \text{ starting from } q.$$

Formally:

$$\delta^*(q, w) := \begin{cases} q & \text{if } w = \varepsilon \\ \delta^*(\delta(q, a), v) & \text{if } w = av \end{cases}$$

Thus: if $w = a_1 \ldots a_n$ and $q \xrightarrow{a_1} q_1 \xrightarrow{a_2} \ldots \xrightarrow{a_n} q_n$, then $\delta^*(q, w) = q_n$

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Example A.11

Pattern matching (Example A.9): on the board

Definition A.12

- $\mathfrak{A}$ accepts $w \in \Sigma^*$ if $\delta^*(q_0, w) \in F$.
- The language recognized (or: accepted) by $\mathfrak{A}$ is

$$L(\mathfrak{A}) := \{w \in \Sigma^* \mid \delta^*(q_0, w) \in F\}.$$

- A language $L \subseteq \Sigma^*$ is called DFA-recognizable if there exists some DFA $\mathfrak{A}$ such that $L(\mathfrak{A}) = L$.
- Two DFA $\mathfrak{A}_1, \mathfrak{A}_2$ are called equivalent if

$$L(\mathfrak{A}_1) = L(\mathfrak{A}_2).$$

Example A.13

1. The set of all bit strings containing 1101 is recognized by the automaton from Example A.9.

Acceptance by DFA III

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2. Two (equivalent) automata recognizing the language

$$\{w \in \mathbb{B}^* \mid w \text{ contains } 1\} :$$

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Example A.13

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   on the board

3. An automaton which recognizes

$$\{w \in \{0, \ldots, 9\}^* \mid \text{value of } w \text{ divisible by } 3\}$$

   Idea: test whether sum of digits is divisible by 3 – one state for each residue class (on the board)

Deterministic Finite Automata

Seen:

- Deterministic finite automata as a model of simple sequential computations
- Recognizability of formal languages by automata

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- Deterministic finite automata as a model of simple sequential computations
- Recognizability of formal languages by automata

Open:

- Composition and transformation of automata?
- Which languages are recognizable, which are not (alternative characterization)?
- Language definition $\mapsto$ automaton and vice versa?

Outline of Part A

1. Formal Languages

2. Finite Automata
   - Deterministic Finite Automata
   - Operations on Languages and Automata
   - Nondeterministic Finite Automata
   - More Decidability Results

3. Regular Expressions

4. Minimization of DFA

5. The Pumping Lemma

6. Outlook

Operations on Languages

Simplest case: Boolean operations (complement, intersection, union)

Question

Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be two DFA with $L(\mathfrak{A}_1) = L_1$ and $L(\mathfrak{A}_2) = L_2$.
Can we construct automata which recognize

- $\overline{L_1}$ ($:= \Sigma^* \setminus L_1$),
- $L_1 \cap L_2$, and
- $L_1 \cup L_2$?

Theorem A.14

If $L \subseteq \Sigma^*$ is DFA-recognizable, then so is $\overline{L}$.

Language Complement

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Proof.

Let $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA such that $L(\mathfrak{A}) = L$. Then:

$$w \in \overline{L} \iff w \notin L \iff \delta^*(q_0, w) \notin F \iff \delta^*(q_0, w) \in Q \setminus F.$$

Thus, $\overline{L}$ is recognized by the DFA $\langle Q, \Sigma, \delta, q_0, Q \setminus F \rangle$. □

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Example A.15

on the board

Language Intersection I

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Proof.

Let $\mathfrak{A}_i = \langle Q_i, \Sigma, \delta_i, q_0^i, F_i \rangle$ be DFA such that $L(\mathfrak{A}_i) = L_i$ ($i = 1, 2$). The new automaton $\mathfrak{A}$ has to accept w iff $\mathfrak{A}_1$ and $\mathfrak{A}_2$ accept w

Idea: let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ run in parallel

- use pairs of states $(q_1, q_2) \in Q_1 \times Q_2$
- start with both components in initial state
- a transition updates both components independently
- for acceptance both components need to be in a final state

$\square$

Language Intersection II

Proof (continued).

Formally: let the product automaton
$$\mathfrak{A} := \langle Q_1 \times Q_2, \Sigma, \delta, (q_0^1, q_0^2), F_1 \times F_2 \rangle$$
be defined by
$$\delta((q_1, q_2), a) := (\delta_1(q_1, a), \delta_2(q_2, a)) \text{ for every } a \in \Sigma.$$

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Proof (continued).

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This definition yields

$$\delta^*((q_1, q_2), w) = (\delta_1^*(q_1, w), \delta_2^*(q_2, w)) \qquad (*)$$

for every $w \in \Sigma^*$.

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for every $w \in \Sigma^*$.

Thus we have:
$$\mathfrak{A} \text{ accepts } w$$
$$\iff \delta^*((q_0^1, q_0^2), w) \in F_1 \times F_2$$
$$\overset{(*)}{\iff} (\delta_1^*(q_0^1, w), \delta_2^*(q_0^2, w)) \in F_1 \times F_2$$
$$\iff \delta_1^*(q_0^1, w) \in F_1 \text{ and } \delta_2^*(q_0^2, w) \in F_2$$
$$\iff \mathfrak{A}_1 \text{ accepts } w \text{ and } \mathfrak{A}_2 \text{ accepts } w \qquad \square$$

Example A.17

on the board

Language Union

Theorem A.18

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Idea: reuse product construction
Construct $\mathfrak{A}$ as before but choose as final states those pairs $(q_1, q_2) \in Q_1 \times Q_2$ with $q_1 \in F_1$ or $q_2 \in F_2$. Thus the set of final states is given by

$$F := (F_1 \times Q_2) \cup (Q_1 \times F_2).$$

$\square$

Language Concatenation

Definition A.19

The concatenation of two languages $L_1, L_2 \subseteq \Sigma^*$ is given by

$$L_1 \cdot L_2 := \{v \cdot w \in \Sigma^* \mid v \in L_1, w \in L_2\}.$$

Abbreviations: $w \cdot L := \{w\} \cdot L$, $L \cdot w := L \cdot \{w\}$

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2. If $L_1 = 00 \cdot \mathbb{B}^*$ and $L_2 = 11 \cdot \mathbb{B}^*$, then
$$L_1 \cdot L_2 = \{w \in \mathbb{B}^* \mid w \text{ has prefix } 00 \text{ and contains } 11\}.$$

Conjecture

If $L_1, L_2 \subseteq \Sigma^*$ are DFA-recognizable, then so is $L_1 \cdot L_2$.

DFA-Recognizability of Concatenation

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Proof (attempt).

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Idea: choose $Q := Q_1 \cup Q_2$ where each $q \in F_1$ is identified with q_0^2

But: on the board $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

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But: on the board $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

Conclusion

Required: automata model where the successor state (for a given state and input symbol) is not unique

Language Iteration

Definition A.21

- The nth power of a language $L \subseteq \Sigma^*$ is the n-fold concatenation of L with itself ($n \in \mathbb{N}$):
$$L^n := \underbrace{L \cdot \ldots \cdot L}_{n \text{ times}} = \{w_1 \ldots w_n \mid \forall i \in \{1, \ldots, n\} : w_i \in L\}.$$
Inductively: $L^0 := \{\varepsilon\}$, $L^{n+1} := L^n \cdot L$

- The iteration (or: Kleene star) of L is
$$L^* := \bigcup_{n \in \mathbb{N}} L^n = \{w_1 \ldots w_n \mid n \in \mathbb{N}, \forall i \in \{1, \ldots, n\} : w_i \in L\}.$$

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Remarks:

- we always have $\varepsilon \in L^*$ (since $L^0 \subseteq L^*$ and $L^0 = \{\varepsilon\}$)
- $w \in L^*$ iff $w = \varepsilon$ or if w can be decomposed into $n \geq 1$ subwords $v_1, \ldots, v_n$ (i.e., $w = v_1 \cdot \ldots \cdot v_n$) such that $v_i \in L$ for every $1 \leq i \leq n$
- again we would suspect that the iteration of a DFA-recognizable language is DFA-recognizable, but there is no simple (deterministic) construction

Operations on Languages and Automata

Seen:

- Operations on languages:
    - complement
    - intersection
    - union
    - concatenation
    - iteration
- DFA constructions for:
    - complement
    - intersection
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Open:

- Automata model for (direct implementation of) concatenation and iteration?

Outline of Part A

Nondeterministic Finite Automata I

Idea:

- for a given state and a given input symbol, several transitions (or none at all) are possible
- an input word generally induces several state sequences ("runs")
- the word is accepted if at least one accepting run exists

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Advantages:

- simplifies representation of languages
  (example: $\mathbb{B}^* \cdot 1101 \cdot \mathbb{B}^*$; on the board)
- yields direct constructions for concatenation and iteration of languages
- more adequate modeling of systems with nondeterministic behaviour
  (communication protocols, multi-agent systems, ...)

Nondeterministic Finite Automata II

Definition A.22

A nondeterministic finite automaton (NFA) is of the form

$$\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$$

where

- Q is a finite set of states
- Σ denotes the input alphabet
- $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of final states

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$$\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$$

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- Q is a finite set of states
- Σ denotes the input alphabet
- $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of final states

Remarks:

- $(q, a, q') \in \Delta$ usually written as $q \xrightarrow{a} q'$
- every DFA can be considered as an NFA
  $((q, a, q') \in \Delta \iff \delta(q, a) = q')$

Acceptance by NFA

Definition A.23

- Let $w = a_1 \ldots a_n \in \Sigma^*$.
- A w-labeled $\mathfrak{A}$-run from q_1 to q_2 is a sequence

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \ldots p_{n-1} \xrightarrow{a_n} p_n$$

  such that $p_0 = q_1$, $p_n = q_2$, and $(p_{i-1}, a_i, p_i) \in \Delta$ for every $1 \leq i \leq n$ (we also write: $q_1 \xrightarrow{w} q_2$).
- $\mathfrak{A}$ accepts w if there is a w-labeled $\mathfrak{A}$-run from q_0 to some $q \in F$
- The language recognized by $\mathfrak{A}$ is
$$L(\mathfrak{A}) := \{w \in \Sigma^* \mid \mathfrak{A} \text{ accepts } w\}.$$
- A language $L \subseteq \Sigma^*$ is called NFA-recognizable if there exists a NFA $\mathfrak{A}$ such that $L(\mathfrak{A}) = L$.
- Two NFA $\mathfrak{A}_1, \mathfrak{A}_2$ are called equivalent if $L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$.

Acceptance Test for NFA

Algorithm A.24 (Acceptance Test for NFA)

Input: NFA $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$, $w \in \Sigma^*$

Question: $w \in L(\mathfrak{A})$?

Procedure: Computation of the *reachability set*
$$R_{\mathfrak{A}}(w) := \{q \in Q \mid q_0 \xrightarrow{w} q\}$$
Iterative procedure for $w = a_1 \ldots a_n$:

1. let $R_{\mathfrak{A}}(\varepsilon) := \{q_0\}$
2. for $i := 1, \ldots, n$: let
   $R_{\mathfrak{A}}(a_1 \ldots a_i) := \{q \in Q \mid \exists p \in R_{\mathfrak{A}}(a_1 \ldots a_{i-1}): p \xrightarrow{a_i} q\}$

Output: "yes" if $R_{\mathfrak{A}}(w) \cap F \neq \emptyset$, otherwise "no"

Remark: this algorithm solves the word problem for NFA

Algorithm A.24 (Acceptance Test for NFA)

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Remark: this algorithm solves the word problem for NFA

Example A.25

on the board

Definition of NFA looks promising, but... (on the board)

Definition of NFA looks promising, but... (on the board)

Solution: admit empty word ε as transition label

ε-**NFA**

Definition A.26

A nondeterministic finite automaton with ε-transitions (ε-NFA) is of the form $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ where

- Q is a finite set of states
- Σ denotes the input alphabet
- $\Delta \subseteq Q \times \Sigma_\varepsilon \times Q$ is the transition relation where $\Sigma_\varepsilon := \Sigma \cup \{\varepsilon\}$
- $q_0 \in Q$ is the initial state
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Remarks:

- every NFA is an ε-NFA
- definitions of runs and acceptance: in analogy to NFA

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- every NFA is an ε-NFA
- definitions of runs and acceptance: in analogy to NFA

Example A.27

on the board

Theorem A.28

If $L_1, L_2 \subseteq \Sigma^*$ are ε-NFA-recognizable, then so is $L_1 \cdot L_2$.

Concatenation and Iteration via ε-NFA

Theorem A.28

If $L_1, L_2 \subseteq \Sigma^*$ are ε-NFA-recognizable, then so is $L_1 \cdot L_2$.

Proof (idea).

on the board $\qquad\square$

Concatenation and Iteration via ε-NFA

Theorem A.28

If $L_1, L_2 \subseteq \Sigma^*$ are ε-NFA-recognizable, then so is $L_1 \cdot L_2$.

Proof (idea).

on the board □

Theorem A.29

If $L \subseteq \Sigma^*$ is ε-NFA-recognizable, then so is L^*.

Proof.

see Theorem A.47 □

Syntax Diagrams as ε-NFA

Syntax diagrams (without recursive calls) can be interpreted as ε-NFA

Example A.30

decimal numbers (on the board)

Types of Finite Automata

1. DFA (Definition A.8)
2. NFA (Definition A.22)
3. ε-NFA (Definition A.26)

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From the definitions we immediately obtain:

Corollary A.31

1. *Every DFA-recognizable language is NFA-recognizable.*
2. *Every NFA-recognizable language is ε-NFA-recognizable.*

Types of Finite Automata

1. DFA (Definition A.8)
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From the definitions we immediately obtain:

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1. *Every DFA-recognizable language is NFA-recognizable.*
2. *Every NFA-recognizable language is ε-NFA-recognizable.*

Goal: establish reverse inclusions

Theorem A.32

Every NFA can be transformed into an equivalent DFA.

From NFA to DFA I

Theorem A.32

Every NFA can be transformed into an equivalent DFA.

Proof.

Idea: let the DFA operate on sets of states ("powerset construction")

- Initial state of DFA := {initial state of NFA}
- $P \xrightarrow{a} P'$ in DFA iff there exist $q \in P, q' \in P'$ such that $q \xrightarrow{a} q'$ in NFA
- P final state in DFA iff it contains some final state of NFA

□

From NFA to DFA II

Proof (continued).

Let $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ be a NFA.

Powerset construction of $\mathfrak{A}' = \langle Q', \Sigma, \delta', q_0', F' \rangle$:

- $Q' := 2^Q := \{ P \mid P \subseteq Q \}$
- $\delta' : Q' \times \Sigma \to Q'$ with
$$q \in \delta'(P, a) \iff \text{there exists } p \in P \text{ such that } (p, a, q) \in \Delta$$
- $q_0' := \{ q_0 \}$
- $F' := \{ P \subseteq Q \mid P \cap F \neq \emptyset \}$

This yields

$$q_0 \xrightarrow{w} q \text{ in } \mathfrak{A} \iff q \in \delta'^*(\{q_0\}, w) \text{ in } \mathfrak{A}'$$

and thus

$$\mathfrak{A} \text{ accepts } w \iff \mathfrak{A}' \text{ accepts } w$$

$\square$

From NFA to DFA II

Proof (continued).

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and thus
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$\square$

Example A.33

on the board

From ε-NFA to NFA

Theorem A.34

Every ε-NFA can be transformed into an equivalent NFA.

From ε-NFA to NFA

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Every ε-NFA can be transformed into an equivalent NFA.

Proof (idea).

Let $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ be a ε-NFA. We construct the NFA $\mathfrak{A}'$ by eliminating all ε-transitions, adding appropriate direct transitions: if $p \xrightarrow{\varepsilon}^* q$, $q \xrightarrow{a} q'$, and $q' \xrightarrow{\varepsilon}^* r$ in $\mathfrak{A}$, then $p \xrightarrow{a} r$ in $\mathfrak{A}'$. Moreover $F' := F \cup \{q_0\}$ if $q_0 \xrightarrow{\varepsilon}^* q \in F$ in $\mathfrak{A}$, and $F' := F$ otherwise. $\square$

From ε-NFA to NFA

Theorem A.34

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Example A.35

on the board

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Example A.35

on the board

Corollary A.36

All types of finite automata recognize the same class of languages.

Seen:

- Definition of ε-NFA
- Determinization of (ε-)NFA

Nondeterministic Finite Automata

Seen:

- Definition of ε-NFA
- Determinization of (ε-)NFA

Open:

- More decidablity results

Outline of Part A

Definition A.37

The word problem for DFA is specified as follows:

Given a DFA $\mathfrak{A}$ and a word $w \in \Sigma^*$, decide whether

$$w \in L(\mathfrak{A}).$$

The Word Problem Revisited

Definition A.37

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Given a DFA $\mathfrak{A}$ and a word $w \in \Sigma^*$, decide whether

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As we have seen (Def. A.10, Alg. A.24, Thm. A.34):

Theorem A.38

The word problem for DFA (NFA, ε-NFA) is decidable.

The Emptiness Problem

Definition A.39

The emptiness problem for DFA is specified as follows:

Given a DFA $\mathfrak{A}$, decide whether $L(\mathfrak{A}) = \emptyset$.

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Proof.

It holds that $L(\mathfrak{A}) \neq \emptyset$ iff in $\mathfrak{A}$ some final state is reachable from the initial state (simple graph-theoretic problem). □

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It holds that $L(\mathfrak{A}) \neq \emptyset$ iff in $\mathfrak{A}$ some final state is reachable from the initial state (simple graph-theoretic problem). $\square$

Remark: important result for formal verification
(unreachability of bad [= final] states)

The Equivalence Problem

Definition A.41

The equivalence problem for DFA is specified as follows:

Given two DFA $\mathfrak{A}_1, \mathfrak{A}_2$, decide whether
$$L(\mathfrak{A}_1) = L(\mathfrak{A}_2).$$

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Theorem A.42

The equivalence problem for DFA (NFA, ε-NFA) is decidable.

Proof.

$\quad L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$

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Proof.

$$\begin{array}{ll} & L(\mathfrak{A}_1) = L(\mathfrak{A}_2) \\ \Longleftrightarrow & L(\mathfrak{A}_1) \subseteq L(\mathfrak{A}_2) \text{ and } L(\mathfrak{A}_2) \subseteq L(\mathfrak{A}_1) \end{array}$$

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Proof.

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\begin{aligned}
& L(\mathfrak{A}_1) = L(\mathfrak{A}_2) \\
\iff\quad & L(\mathfrak{A}_1) \subseteq L(\mathfrak{A}_2) \text{ and } L(\mathfrak{A}_2) \subseteq L(\mathfrak{A}_1) \\
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\end{aligned}
$$

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\Longleftrightarrow\ & (L(\mathfrak{A}_1) \setminus L(\mathfrak{A}_2)) \cup (L(\mathfrak{A}_2) \setminus L(\mathfrak{A}_1)) = \emptyset \\
\Longleftrightarrow\ & (L(\mathfrak{A}_1) \cap \underbrace{\overline{L(\mathfrak{A}_2)}}_{\text{DFA-recognizable (Thm. A.14)}}) \cup (L(\mathfrak{A}_2) \cap \underbrace{\overline{L(\mathfrak{A}_1)}}_{\text{DFA-recognizable (Thm. A.14)}}) = \emptyset
\end{aligned}
$$

DFA-recognizable (Thm. A.16)   DFA-recognizable (Thm. A.16)

DFA-recognizable (Thm. A.18)

decidable (Thm. A.40)

□

Finite Automata

Seen:

- Decidability of word problem
- Decidability of emptiness problem
- Decidability of equivalence problem

Finite Automata

Seen:

- Decidability of word problem
- Decidability of emptiness problem
- Decidability of equivalence problem

Open:

- Non-algorithmic description of languages

Outline of Part A

1. Formal Languages

2. Finite Automata
   - Deterministic Finite Automata
   - Operations on Languages and Automata
   - Nondeterministic Finite Automata
   - More Decidability Results

3. Regular Expressions

4. Minimization of DFA

5. The Pumping Lemma

6. Outlook

An Example

Example A.43

Consider the set of all words over $\Sigma := \{a, b\}$ which

1. start with one or three a symbols
2. continue with a (potentially empty) sequence of blocks, each containing at least one b and exactly two a's
3. conclude with a (potentially empty) sequence of b's

Corresponding regular expression:

$$(a + aaa)(\underbrace{bb^*ab^*ab^*}_{b \text{ before } a\text{'s}} + \underbrace{b^*abb^*ab^*}_{b \text{ between } a\text{'s}} + \underbrace{b^*ab^*abb^*}_{b \text{ after } a\text{'s}})^*b^*$$

Definition A.44

The set of regular expressions over Σ is inductively defined by:

- $\emptyset$ and ε are regular expressions
- every $a \in \Sigma$ is a regular expression
- if α and β are regular expressions, then so are
  - $\alpha + \beta$
  - $\alpha \cdot \beta$
  - α^*

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  - $\alpha + \beta$
  - $\alpha \cdot \beta$
  - α^*

Notation:

- $\cdot$ can be omitted
- * binds stronger than $\cdot$, $\cdot$ binds stronger than $+$
- α^+ abbreviates $\alpha \cdot \alpha^*$

Semantics of Regular Expressions

Definition A.45

Every regular expression α defines a language $L(\alpha)$:

$$
\begin{aligned}
L(\emptyset) &:= \emptyset \\
L(\varepsilon) &:= \{\varepsilon\} \\
L(a) &:= \{a\} \\
L(\alpha + \beta) &:= L(\alpha) \cup L(\beta) \\
L(\alpha \cdot \beta) &:= L(\alpha) \cdot L(\beta) \\
L(\alpha^*) &:= (L(\alpha))^*
\end{aligned}
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Every regular expression α defines a language $L(\alpha)$:

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L(\alpha + \beta) &:= L(\alpha) \cup L(\beta) \\
L(\alpha \cdot \beta) &:= L(\alpha) \cdot L(\beta) \\
L(\alpha^*) &:= (L(\alpha))^*
\end{aligned}
$$

A language L is called regular if it is definable by a regular expression, i.e., if $L = L(\alpha)$ for some regular expression α.

Regular Languages

Example A.46

1. $\{aa\}$ is regular since

$$L(a \cdot a) = L(a) \cdot L(a) = \{a\} \cdot \{a\} = \{aa\}$$

Regular Languages

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$$L((a + b)^*) = (L(a + b))^* = (L(a) \cup L(b))^* = (\{a\} \cup \{b\})^* = \{a, b\}^*$$

Regular Languages

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1. $\{aa\}$ is regular since

$$L(a \cdot a) = L(a) \cdot L(a) = \{a\} \cdot \{a\} = \{aa\}$$

2. $\{a, b\}^*$ is regular since

$$L((a + b)^*) = (L(a + b))^* = (L(a) \cup L(b))^* = (\{a\} \cup \{b\})^* = \{a, b\}^*$$

3. The set of all words over $\{a, b\}$ containing abb is regular since

$$L((a + b)^* \cdot a \cdot b \cdot b \cdot (a + b)^*) = \{a, b\}^* \cdot \{abb\} \cdot \{a, b\}^*$$

Theorem A.47 (Kleene's Theorem)

To each regular expression there corresponds an ε-NFA, and vice versa.

Regular Languages and Finite Automata I

Theorem A.47 (Kleene's Theorem)

To each regular expression there corresponds an ε-NFA, and vice versa.

Proof.

$\implies$ using induction over the given regular expression α, we construct an ε-NFA $\mathfrak{A}_\alpha$

- with exactly one final state q_f
- without transitions into the initial state
- without transitions leaving the final state

(on the board)

$\impliedby$ by solving a regular equation system (details omitted)

$\square$

Corollary A.48

The following properties are equivalent:

- *L is regular*
- *L is DFA-recognizable*
- *L is NFA-recognizable*
- *L is ε-NFA-recognizable*

Implementation of Pattern Matching

Algorithm A.49 (Pattern Matching)

Input: *regular expression α and $w \in \Sigma^*$*

Question: *does w contain some $v \in L(\alpha)$?*

Procedure:
1. *let $\beta := (a_1 + \ldots + a_n)^* \cdot \alpha$ (for $\Sigma = \{a_1, \ldots, a_n\}$)*
2. *determine ε-NFA $\mathfrak{A}_\beta$ for β*
3. *eliminate ε-transitions*
4. *apply powerset construction to obtain DFA $\mathfrak{A}$*
5. *let $\mathfrak{A}$ run on w*

Output: *"yes" if $\mathfrak{A}$ passes through some final state, otherwise "no"*

Remark: in UNIX/LINUX implemented by `grep` and `lex`

Regular Expressions in UNIX (`grep`, `flex`, ...)

Syntax	Meaning
printable character	this character
\n, \t, \123, etc.	newline, tab, octal representation, etc.
.	any character except \n
[*Chars*]	one of *Chars*; ranges possible ("0–9")
[^*Chars*]	none of *Chars*
\\, \., \[, etc.	\, ., [, etc.
"*Text*"	*Text* without interpretation of ., [, \, etc.
^α	α at beginning of line
α\$	α at end of line
α?	zero or one α
α*	zero or more α
α+	one or more α
α\{*n, m*\}	between *n* and *m* times α (",*m*" optional)
(α)	α
$\alpha_1 \alpha_2$	concatenation
$\alpha_1 \mid \alpha_2$	alternative

Regular Expressions

Seen:

- Definition of regular expressions
- Equivalence of regular and DFA-recognizable languages

Outline of Part A

Goal: space-efficient implementation of regular languages

Given: DFA $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$

Wanted: DFA $\mathfrak{A}_{min} = \langle Q', \Sigma, \delta', q_0', F' \rangle$ such that $L(\mathfrak{A}_{min}) = L(\mathfrak{A})$ and $|Q'|$ minimal

State Equivalence

Example A.50

NFA for accepting $(a + b)^* ab(a + b)^*$:
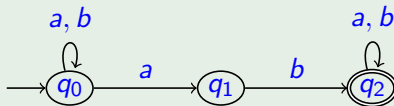
State Equivalence

Example A.50

NFA for accepting $(a + b)^* ab(a + b)^*$:
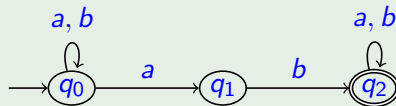
Powerset construction yields DFA $\mathfrak{A}$:
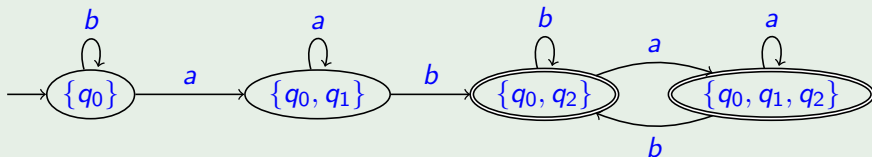
Example A.50

NFA for accepting $(a + b)^* ab(a + b)^*$:
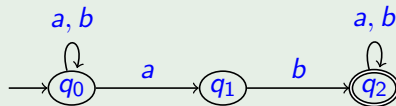
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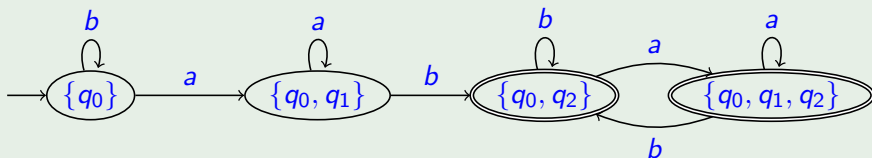
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State Equivalence

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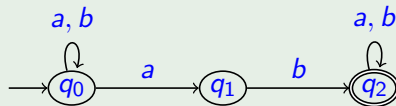
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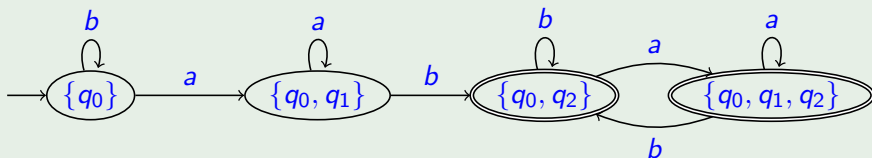
Observation: $\{q_0, q_2\}$ and $\{q_0, q_1, q_2\}$ are equivalent

Definition A.51

Given DFA $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$, states $p, q \in Q$ are equivalent if
$$\forall w \in \Sigma^* : \delta^*(p, w) \in F \iff \delta^*(q, w) \in F.$$

Minimization

Minimization: merging of equivalent states

Example A.52 (cf. Example A.50)

DFA after state merging:
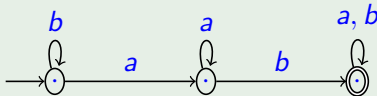
Minimization

Minimization: merging of equivalent states

Example A.52 (cf. Example A.50)

DFA after state merging:
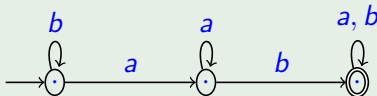
Problem: identification of equivalent states

Approach: iterative computation of inequivalent states by refinement

Corollary A.53

$p, q \in Q$ are *inequivalent* if there exists $w \in \Sigma^*$ such that
$$\delta^*(p, w) \in F \text{ and } \delta^*(q, w) \notin F$$
(or vice versa, i.e., p and q can be distinguished by w)

Lemma A.54

Inductive characterization of state inequivalence:

- $w = \varepsilon$: $p \in F, q \notin F \implies p, q$ *inequivalent (by ε)*
- $w = av$: p', q' *inequivalent (by v),* $p \xrightarrow{a} p', q \xrightarrow{a} q'$
  $\implies p, q$ *inequivalent (by w)*

Computing State (In-)Equivalence

Lemma A.54

Inductive characterization of state inequivalence:

- $w = \varepsilon$: $p \in F, q \notin F \implies p, q$ inequivalent (by ε)
- $w = av$: p', q' inequivalent (by v), $p \xrightarrow{a} p', q \xrightarrow{a} q'$
  $\implies p, q$ inequivalent (by w)

Algorithm A.55 (State Equivalence for DFA)

Input: DFA $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$

Procedure: Computation of "equivalence matrix" over $Q \times Q$

1. mark every pair (p, q) with $p \in F, q \notin F$ by ε
2. for every unmarked pair (p, q) and every $a \in \Sigma$:
   if $(\delta(p, a), \delta(q, a))$ marked by v, then mark (p, q) by av
3. repeat until no change

Output: all equivalent ($=$ unmarked) pairs of states

Example A.56

Given DFA:
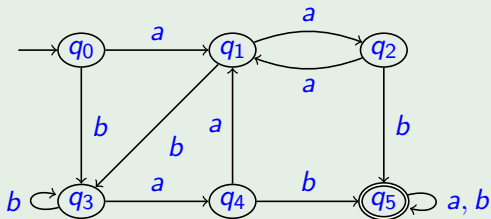
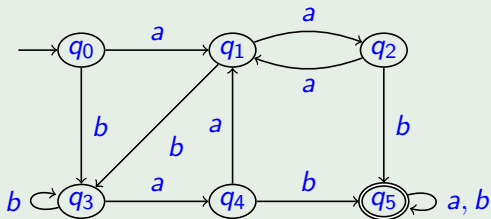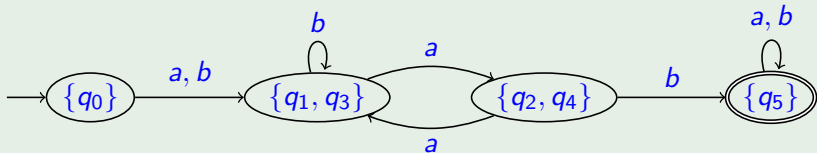
Equivalence matrix: on the board

Minimization Example

Example A.56

Given DFA:

Equivalence matrix: on the board
Resulting minimal DFA:

Correctness of Minimization

Theorem A.57

For every DFA $\mathfrak{A}$,
$$L(\mathfrak{A}) = L(\mathfrak{A}_{min})$$

Correctness of Minimization

Theorem A.57

For every DFA $\mathfrak{A}$,

$$L(\mathfrak{A}) = L(\mathfrak{A}_{min})$$

Remark: the minimal DFA is unique, in the following sense:

$$\forall \text{DFA } \mathfrak{A}, \mathfrak{B} : L(\mathfrak{A}) = L(\mathfrak{B}) \implies \mathfrak{A}_{min} \approx \mathfrak{B}_{min}$$

where $\approx$ refers to automata isomorphism ($=$ identity up to naming of states)

Outline of Part A

Observation: a language L is DFA-recognizable (and thus regular) if the membership of a word w can be tested by symbol-wise reading of w, using a bounded memory

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Conjecture: languages of the form $\{a^n b^n \mid n \in \mathbb{N}\}$ are not regular since the test for membership requires the capability of comparing the number of a symbols to the number of b symbols (which can grow arbitrarily large)

The Pumping Lemma I

Theorem A.58 (Pumping Lemma for Regular Languages)

If L is regular, then there exists $n \geq 1$ (called *pumping index*) such that any $w \in L$ with $|w| \geq n$ can be decomposed as $w = xyz$ where

- $y \neq \varepsilon$ and
- for every $i \geq 0$, $xy^i z \in L$

Proof (idea).

Let $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA such that $L(\mathfrak{A}) = L$. Choose $n := |Q|$, and let $w \in L$.

Then:
$\qquad w = a_1 \ldots a_k$ with $k \geq n$

$\implies$ the accepting run visits $k + 1 \geq n + 1$ states:
$\qquad q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \ldots \xrightarrow{a_k} q_k$

$\implies$ some state in Q occurs (at least) twice:
$\qquad$ there exist $1 \leq i < j \leq k$ such that $q_i = q_j$

Choose $y := a_{i+1} \ldots a_j$ to be the substring which is read between the two visits of q. Clearly, $y \neq \varepsilon$. Moreover the cycle can be omitted or repeated such that $xz \in L$, $xyz \in L$, $xy^2 z \in L$, ... $\qquad \square$

The Pumping Lemma II

Remark: Pumping Lemma states a necessary condition for regularity
$\implies$ can only be used to show the non-regularity of a language

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Example A.59

1. $L := \{a^k b^k \mid k \in \mathbb{N}\}$ is not regular. Proof by contradiction: Assume that L is regular, and let n be a pumping index. Consider $w := a^n b^n$. Since $|w| \geq n$, it can be decomposed as $w = xyz$ with $y \neq \varepsilon$. The following cases are possible:
   - $y \in L(a^+)$: then $xy^2z \notin L$ (more as than bs)
   - $y \in L(b^+)$: then $xy^2z \notin L$ (less as than bs)
   - $y \in L(a^+b^+)$: then $xy^2z \notin L$ (a follows b)

The Pumping Lemma II

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2. Similarly: the set of all arithmetic expressions is not regular

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---

Conclusion

Finite automata are too weak for defining the syntax of programming languages (consider $a = $ "(", $b = $ ")")!

Seen:

- Necessary condition for regularity of languages
- Counterexamples

The Pumping Lemma IV

Seen:

- Necessary condition for regularity of languages
- Counterexamples

Open:

- More expressive formalisms for describing languages?

Outline of Part A

Outlook

- More language operations (homomorphisms, ...)
- Construction of scanners for compilers