Compiler Construction

Lecture 5: Syntactic Analysis I (Introduction)

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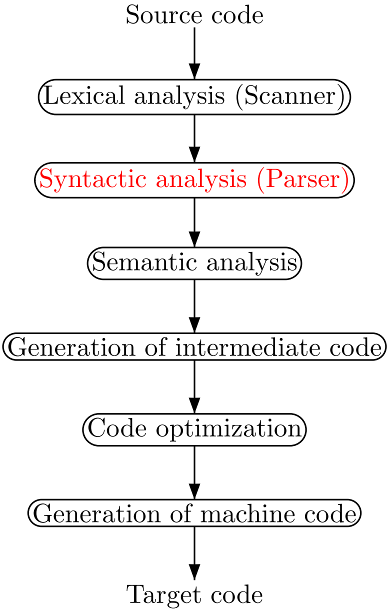
Summer semester 2008

Conceptual Structure of a Compiler



Outline

Syntactic Structures

Starting point: sequence of symbols as produced by the scanner
Here: ignore attribute information

- Σ (finite) set of tokens (= syntactic atoms; terminals)
  (e.g., $\{\mathsf{id}, \mathsf{if}, \mathsf{int}, \ldots\}$)
- $w \in \Sigma^*$ token sequence
  (of course, not every $w \in \Sigma^*$ forms a valid program)

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Observation: the hierarchical structure of syntactic units can be described by context-free grammars

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The goal of syntactic analysis is to determine the syntactic structure of a program, given by a token sequence, according to a context-free grammar.

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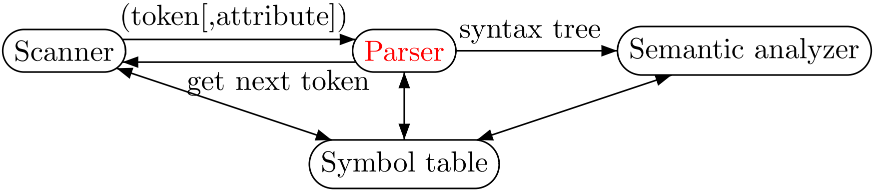

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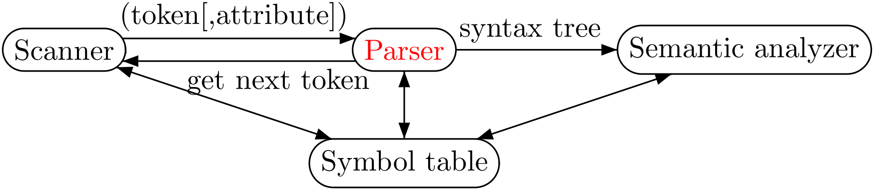
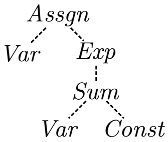


Example:

$\ldots(\mathsf{id}, p_1)(\mathsf{gets},)(\mathsf{id}, p_2)(\mathsf{plus},)(\mathsf{int}, 1)(\mathsf{sem},)\ldots \rightsquigarrow$



Outline

Context-Free Grammars I

Definition 5.2 (Syntax of context-free grammars)

A context-free grammar (CFG) (over Σ) is a quadruple

$$G = \langle N, \Sigma, P, S \rangle$$

where

- N is a finite set of nonterminal symbols,
- Σ is a (finite) alphabet of terminal symbols (disjoint from N),
- P is a finite set of production rules of the form $A \to \alpha$ where $A \in N$ and $\alpha \in X^*$ for $X := N \cup \Sigma$, and
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Remarks: as denotations we generally use

- $A, B, C, \ldots \in N$ for nonterminal symbols
- $a, b, c, \ldots \in \Sigma$ for terminal symbols
- $u, v, w, \ldots \in \Sigma^*$ for terminal words
- $\alpha, \beta, \gamma, \ldots \in X^*$ for sentences

Context-Free Grammars II

Context-free grammars generate context-free languages:

Definition 5.3 (Semantics of context-free grammars)

Let $G = \langle N, \Sigma, P, S \rangle$ be a context-free grammar.

- The derivation relation $\Rightarrow \subseteq X^* \times X^*$ of G is defined by

$$\alpha \Rightarrow \beta \text{ iff there exist } \alpha_1, \alpha_2 \in X^*, A \rightarrow \gamma \in P$$
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Remark: obviously,

$$L(G) = \{w \in \Sigma^* \mid S \Rightarrow_l^* w\} = \{w \in \Sigma^* \mid S \Rightarrow_r^* w\}$$

Example 5.4

The grammar $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$ over $\Sigma := \{a, b\}$, given by the productions

$$S \to aSb \mid \varepsilon,$$

generates the context-free (and non-regular) language

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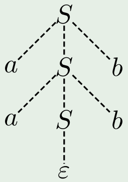
The example derivation

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

can be represented by the following syntax tree for $aabb$:



Syntax Trees, Derivations, and Words

Remark: the connection between derivations, syntax trees, and generated words is not unique

1. A syntax tree generally represents several derivations.
2. A derivation can generally be represented by several syntax trees.
3. A word can generally be produced by several derivations.
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However:

1. Every syntax tree yields exactly one word (= concatenation of leafs).
2. Every syntax tree corresponds to exactly one leftmost derivation, and vice versa.
3. Every syntax tree corresponds to exactly one rightmost derivation, and vice versa.

(Un-)Ambiguity of CFGs and CFLs

Definition 5.6 (Ambiguity)

- A context-free grammar $G \in CFG_\Sigma$ is called unambiguous if every word $w \in L(G)$ has exactly one syntax tree. Otherwise it is called ambiguous.

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Example 5.7

on the board

Corollary 5.8

A grammar $G \in CFG_\Sigma$ is unambiguous
iff every word $w \in L(G)$ has exactly one leftmost derivation
iff every word $w \in L(G)$ has exactly one rightmost derivation.

Outline

Parsing Context-Free Languages

Problem 5.9 (Word problem for context-free languages)

Given $G \in CFG_\Sigma$ *and* $w \in \Sigma^*$, *decide whether* $w \in L(G)$ *(and determine a corresponding syntax tree).*

Parsing Context-Free Languages

Problem 5.9 (Word problem for context-free languages)

Given $G \in CFG_\Sigma$ and $w \in \Sigma^*$, decide whether $w \in L(G)$ (and determine a corresponding syntax tree).

- Decidable for arbitrary CFGs (in Chomsky Normal Form) using the tabular method by Cocke, Younger, and Kasami ("CYK Algorithm"; space/time complexity $\mathcal{O}(|w|^2)/\mathcal{O}(|w|^3)$)

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Two approaches:

Top-down analysis: construction of syntax tree from the root towards the leafs, representation as leftmost derivation

Bottom-up analysis: construction of syntax tree from the leafs towards the root, representation as (reversed) rightmost derivation

Leftmost/Rightmost Analysis I

Goal: compact representation of left-/rightmost derivations by index sequences

Definition 5.10 (Leftmost/rightmost analysis)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$, $p := |P|$, and $\pi : [p] \to P$ a bijection.

- If $i \in [p]$, $\pi(i) = A \to \gamma$, $w \in \Sigma^*$, and $\alpha \in X^*$, then we write
$$wA\alpha \overset{i}{\Rightarrow}_l w\gamma\alpha \quad \text{and} \quad \alpha A w \overset{i}{\Rightarrow}_r \alpha\gamma w.$$

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- An index sequence $z \in [p]^*$ is called a leftmost analysis (rightmost analysis) of α if $S \overset{z}{\Rightarrow}_l \alpha$ ($S \overset{z}{\Rightarrow}_r \alpha$), respectively.

Example 5.11

Grammar for arithmetic expressions:

$$G_{AE}: \begin{array}{rcll} E & \to & E\texttt{+}T \mid T & (1,2) \\ T & \to & T\texttt{*}F \mid F & (3,4) \\ F & \to & \texttt{(}E\texttt{)} \mid \texttt{a} \mid \texttt{b} & (5,6,7) \end{array}$$

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\end{array}
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Leftmost derivation of `(a)*b`:

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\begin{array}{lllllllll}
E & \stackrel{2}{\Rightarrow}_l & T & \stackrel{3}{\Rightarrow}_l & T\texttt{*}F & \stackrel{4}{\Rightarrow}_l & F\texttt{*}F & \stackrel{5}{\Rightarrow}_l & \texttt{(}E\texttt{)*}F \\
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$\Longrightarrow$ leftmost analysis: 23452467

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Rightmost derivation of `(a)*b`:

$$\begin{array}{ccccccccc} E & \overset{2}{\Rightarrow}_r & T & \overset{3}{\Rightarrow}_r & T\texttt{*}F & \overset{7}{\Rightarrow}_r & T\texttt{*b} & \overset{4}{\Rightarrow}_r & F\texttt{*b} \\ & \overset{5}{\Rightarrow}_r & \texttt{(}E\texttt{)*b} & \overset{2}{\Rightarrow}_r & \texttt{(}T\texttt{)*b} & \overset{4}{\Rightarrow}_r & \texttt{(}F\texttt{)*b} & \overset{6}{\Rightarrow}_r & \texttt{(a)*b} \end{array}$$

$\Longrightarrow$ rightmost analysis: 23745246

Reducedness of Context-Free Grammars

General assumption in the following: every grammar is reduced

Definition 5.12 (Reduced CFG)

A grammar $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$ is called reduced if for every $A \in N$ there exist $\alpha, \beta \in X^*$ and $w \in \Sigma^*$ such that

$$\begin{aligned} S &\Rightarrow^* \alpha A \beta \quad && (A \text{ reachable}) \text{ and} \\ A &\Rightarrow^* w && (A \text{ productive}). \end{aligned}$$

Outline

Approach:

1. Given $G \in CFG_\Sigma$, construct a nondeterministic pushdown automaton (PDA) which accepts $L(G)$ and which additionally computes corresponding leftmost derivations (similar to the proof of "$L(CFG_\Sigma) \subseteq L(PDA_\Sigma)$")
    - input alphabet: Σ
    - pushdown alphabet: X
    - output alphabet: $[p]$
    - state set: omitted

Top-Down Parsing

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   - input alphabet: Σ
   - pushdown alphabet: X
   - output alphabet: $[p]$
   - state set: omitted
2. Remove nondeterminism by allowing lookahead on the input: $G \in LL(k)$ iff $L(G)$ recognizable by deterministic PDA with lookahead of k symbols

Definition 5.13 (Nondeterministic top-down parsing automaton)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$. The nondeterministic top-down parsing automaton of G, NTA(G), is defined by the following components.

- Input alphabet: Σ
- Pushdown alphabet: X
- Output alphabet: $[p]$
- Configurations: $\Sigma^* \times X^* \times [p]^*$ (top of pushdown to the left)
- Transitions for $w \in \Sigma^*$, $\alpha \in X^*$, and $z \in [p]^*$:
  expansion steps: if $\pi(i) = A \to \beta$, then $(w, A\alpha, z) \vdash (w, \beta\alpha, zi)$
  matching steps: for every $a \in \Sigma$, $(aw, a\alpha, z) \vdash (w, \alpha, z)$
- Initial configuration for $w \in \Sigma^*$: (w, S, ε)
- Final configurations: $\{\varepsilon\} \times \{\varepsilon\} \times [p]^*$

The Nondeterministic Top-Down Automaton I

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- Final configurations: $\{\varepsilon\} \times \{\varepsilon\} \times [p]^*$

Remark: NTA(G) is nondeterministic iff G contains $A \to \beta \mid \gamma$

The Nondeterministic Top-Down Automaton II

Example 5.14

Grammar for
arithmetic expressions
(cf. Example 5.11):

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Leftmost analysis of `(a)*b`:

$$(\texttt{(a)*b}, E, \varepsilon)$$

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\vdash\ & (\texttt{(a)*b}, T\texttt{*}F, 23)
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Leftmost analysis of `(a)*b`:

$$
\begin{aligned}
&(\texttt{(a)*b}, E, \varepsilon)\\
\vdash\ &(\texttt{(a)*b}, T, 2)\\
\vdash\ &(\texttt{(a)*b}, T\texttt{*}F, 23)\\
\vdash\ &(\texttt{(a)*b}, F\texttt{*}F, 234)
\end{aligned}
$$

Example 5.14

Grammar for arithmetic expressions (cf. Example 5.11):

$$
\begin{aligned}
G_{AE}: E &\to E\texttt{+}T \mid T & (1,2)\\
T &\to T\texttt{*}F \mid F & (3,4)\\
F &\to \texttt{(}E\texttt{)} \mid \texttt{a} \mid \texttt{b} & (5,6,7)
\end{aligned}
$$

Leftmost analysis of `(a)*b`:

$$
\begin{aligned}
& (\texttt{(a)*b}, E, \varepsilon)\\
\vdash\ & (\texttt{(a)*b}, T, 2)\\
\vdash\ & (\texttt{(a)*b}, T\texttt{*}F, 23)\\
\vdash\ & (\texttt{(a)*b}, F\texttt{*}F, 234)\\
\vdash\ & (\texttt{(a)*b}, \texttt{(}E\texttt{)*}F, 2345)
\end{aligned}
$$

Example 5.14

Grammar for arithmetic expressions (cf. Example 5.11):

$$
\begin{array}{llll}
G_{AE}: & E \rightarrow E\texttt{+}T \mid T & & (1,2) \\
& T \rightarrow T\texttt{*}F \mid F & & (3,4) \\
& F \rightarrow \texttt{(}E\texttt{)} \mid \texttt{a} \mid \texttt{b} & & (5,6,7)
\end{array}
$$

Leftmost analysis of `(a)*b`:

$$
\begin{array}{ll}
& (\texttt{(a)*b}, E, \varepsilon) \\
\vdash & (\texttt{(a)*b}, T, 2) \\
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\vdash & (\texttt{(a)*b}, F\texttt{*}F, 234) \\
\vdash & (\texttt{(a)*b}, \texttt{(}E\texttt{)*}F, 2345) \\
\vdash & (\texttt{a)*b}, E\texttt{)*}F, 2345)
\end{array}
$$

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$$
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Leftmost analysis of `(a)*b`:

$$
\begin{array}{ll}
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\vdash & (\texttt{(a)*b}, F\texttt{*}F, 234) \\
\vdash & (\texttt{(a)*b}, \texttt{(}E\texttt{)*}F, 2345) \\
\vdash & (\texttt{a)*b}, E\texttt{)*}F, 2345) \\
\vdash & (\texttt{a)*b}, T\texttt{)*}F, 23452)
\end{array}
$$

Example 5.14

Grammar for arithmetic expressions (cf. Example 5.11):

$$
\begin{array}{llll}
G_{AE}: & E \rightarrow E\texttt{+}T \mid T & (1,2) \\
& T \rightarrow T\texttt{*}F \mid F & (3,4) \\
& F \rightarrow \texttt{(}E\texttt{)} \mid \texttt{a} \mid \texttt{b} & (5,6,7)
\end{array}
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Leftmost analysis of `(a)*b`:

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\vdash & (\texttt{(a)*b}, \texttt{(}E\texttt{)*}F, 2345) \\
\vdash & (\texttt{a)*b}, E\texttt{)*}F, 2345) \\
\vdash & (\texttt{a)*b}, T\texttt{)*}F, 23452) \\
\vdash & (\texttt{a)*b}, F\texttt{)*}F, 234524)
\end{array}
$$

Example 5.14

Grammar for arithmetic expressions (cf. Example 5.11):

$$
\begin{array}{llll}
G_{AE}: & E \to E\texttt{+}T \mid T & (1,2) \\
 & T \to T\texttt{*}F \mid F & (3,4) \\
 & F \to \texttt{(}E\texttt{)} \mid \texttt{a} \mid \texttt{b} & (5,6,7)
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Leftmost analysis of `(a)*b`:

$$
\begin{array}{lll}
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\vdash & (\texttt{(a)*b}, F\texttt{*}F, 234) \\
\vdash & (\texttt{(a)*b}, \texttt{(}E\texttt{)*}F, 2345) \\
\vdash & (\texttt{a)*b}, E\texttt{)*}F, 2345) \\
\vdash & (\texttt{a)*b}, T\texttt{)*}F, 23452) \\
\vdash & (\texttt{a)*b}, F\texttt{)*}F, 234524) \\
\vdash & (\texttt{a)*b}, \texttt{a)*}F, 2345246)
\end{array}
$$

Example 5.14

Grammar for arithmetic expressions (cf. Example 5.11):

$$
\begin{array}{rlll}
G_{AE}: & E \rightarrow E\texttt{+}T \mid T & & (1,2) \\
 & T \rightarrow T\texttt{*}F \mid F & & (3,4) \\
 & F \rightarrow \texttt{(}E\texttt{)} \mid \texttt{a} \mid \texttt{b} & & (5,6,7)
\end{array}
$$

Leftmost analysis of `(a)*b`:

$$
\begin{array}{ll}
 & (\texttt{(a)*b}, E, \varepsilon) \\
\vdash & (\texttt{(a)*b}, T, 2) \\
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\vdash & (\texttt{(a)*b}, F\texttt{*}F, 234) \\
\vdash & (\texttt{(a)*b}, \texttt{(}E\texttt{)*}F, 2345) \\
\vdash & (\texttt{a)*b}, E\texttt{)*}F, 2345) \\
\vdash & (\texttt{a)*b}, T\texttt{)*}F, 23452) \\
\vdash & (\texttt{a)*b}, F\texttt{)*}F, 234524) \\
\vdash & (\texttt{a)*b}, \texttt{a)*}F, 2345246) \\
\vdash & (\texttt{)*b}, \texttt{)*}F, 2345246)
\end{array}
$$

Example 5.14

Grammar for arithmetic expressions (cf. Example 5.11):

$$
\begin{array}{llll}
G_{AE}: & E \rightarrow E\texttt{+}T \mid T & (1,2) \\
 & T \rightarrow T\texttt{*}F \mid F & (3,4) \\
 & F \rightarrow \texttt{(}E\texttt{)} \mid \texttt{a} \mid \texttt{b} & (5,6,7)
\end{array}
$$

Leftmost analysis of `(a)*b`:

$$
\begin{array}{llll}
 & (\texttt{(a)*b}, & E & , \varepsilon \;) \\
\vdash & (\texttt{(a)*b}, & T & , 2 \;) \\
\vdash & (\texttt{(a)*b}, & T\texttt{*}F & , 23 \;) \\
\vdash & (\texttt{(a)*b}, & F\texttt{*}F & , 234 \;) \\
\vdash & (\texttt{(a)*b}, & \texttt{(}E\texttt{)*}F & , 2345 \;) \\
\vdash & (\texttt{a)*b}, & E\texttt{)*}F & , 2345 \;) \\
\vdash & (\texttt{a)*b}, & T\texttt{)*}F & , 23452 \;) \\
\vdash & (\texttt{a)*b}, & F\texttt{)*}F & , 234524 \;) \\
\vdash & (\texttt{a)*b}, & \texttt{a)*}F & , 2345246 \;) \\
\vdash & (\texttt{)*b}, & \texttt{)*}F & , 2345246 \;) \\
\vdash & (\texttt{*b}, & \texttt{*}F & , 2345246 \;)
\end{array}
$$

Example 5.14

Grammar for arithmetic expressions (cf. Example 5.11):

$$
\begin{array}{llll}
G_{AE}: & E \rightarrow E\texttt{+}T \mid T & & (1,2) \\
 & T \rightarrow T\texttt{*}F \mid F & & (3,4) \\
 & F \rightarrow \texttt{(}E\texttt{)} \mid \texttt{a} \mid \texttt{b} & & (5,6,7)
\end{array}
$$

Leftmost analysis of `(a)*b`:

$$
\begin{array}{lllll}
 & (\texttt{(a)*b}, & E & , \varepsilon &) \\
\vdash & (\texttt{(a)*b}, & T & , 2 &) \\
\vdash & (\texttt{(a)*b}, & T\texttt{*}F & , 23 &) \\
\vdash & (\texttt{(a)*b}, & F\texttt{*}F & , 234 &) \\
\vdash & (\texttt{(a)*b}, & \texttt{(}E\texttt{)*}F & , 2345 &) \\
\vdash & (\texttt{a)*b}, & E\texttt{)*}F & , 2345 &) \\
\vdash & (\texttt{a)*b}, & T\texttt{)*}F & , 23452 &) \\
\vdash & (\texttt{a)*b}, & F\texttt{)*}F & , 234524 &) \\
\vdash & (\texttt{a)*b}, & \texttt{a)*}F & , 2345246 &) \\
\vdash & (\texttt{)*b}, & \texttt{)*}F & , 2345246 &) \\
\vdash & (\texttt{*b}, & \texttt{*}F & , 2345246 &) \\
\vdash & (\texttt{b}, & F & , 2345246 &)
\end{array}
$$

Example 5.14

Grammar for arithmetic expressions (cf. Example 5.11):

$$\begin{array}{llll} G_{AE}: & E \rightarrow E\texttt{+}T \mid T & (1,2) \\ & T \rightarrow T\texttt{*}F \mid F & (3,4) \\ & F \rightarrow \texttt{(}E\texttt{)} \mid \texttt{a} \mid \texttt{b} & (5,6,7) \end{array}$$

Leftmost analysis of `(a)*b`:

$$\begin{array}{lll} & (\texttt{(a)*b}, E & , \varepsilon) \\ \vdash & (\texttt{(a)*b}, T & , 2) \\ \vdash & (\texttt{(a)*b}, T\texttt{*}F & , 23) \\ \vdash & (\texttt{(a)*b}, F\texttt{*}F & , 234) \\ \vdash & (\texttt{(a)*b}, \texttt{(}E\texttt{)*}F & , 2345) \\ \vdash & (\texttt{a)*b}, E\texttt{)*}F & , 2345) \\ \vdash & (\texttt{a)*b}, T\texttt{)*}F & , 23452) \\ \vdash & (\texttt{a)*b}, F\texttt{)*}F & , 234524) \\ \vdash & (\texttt{a)*b}, \texttt{a)*}F & , 2345246) \\ \vdash & (\texttt{)*b}, \texttt{)*}F & , 2345246) \\ \vdash & (\texttt{*b}, \texttt{*}F & , 2345246) \\ \vdash & (\texttt{b}, F & , 2345246) \\ \vdash & (\texttt{b}, \texttt{b} & , 23452467) \end{array}$$

Example 5.14

Grammar for arithmetic expressions (cf. Example 5.11):

$$
\begin{array}{rll}
G_{AE}: E & \rightarrow E\texttt{+}T \mid T & (1,2) \\
T & \rightarrow T\texttt{*}F \mid F & (3,4) \\
F & \rightarrow \texttt{(}E\texttt{)} \mid \texttt{a} \mid \texttt{b} & (5,6,7)
\end{array}
$$

Leftmost analysis of `(a)*b`:

$$
\begin{array}{lll}
& (\texttt{(a)*b}, E & , \varepsilon) \\
\vdash & (\texttt{(a)*b}, T & , 2) \\
\vdash & (\texttt{(a)*b}, T\texttt{*}F & , 23) \\
\vdash & (\texttt{(a)*b}, F\texttt{*}F & , 234) \\
\vdash & (\texttt{(a)*b}, \texttt{(}E\texttt{)*}F & , 2345) \\
\vdash & (\texttt{a)*b}, E\texttt{)*}F & , 2345) \\
\vdash & (\texttt{a)*b}, T\texttt{)*}F & , 23452) \\
\vdash & (\texttt{a)*b}, F\texttt{)*}F & , 234524) \\
\vdash & (\texttt{a)*b}, \texttt{a)*}F & , 2345246) \\
\vdash & (\texttt{)*b}, \texttt{)*}F & , 2345246) \\
\vdash & (\texttt{*b}, \texttt{*}F & , 2345246) \\
\vdash & (\texttt{b}, F & , 2345246) \\
\vdash & (\texttt{b}, \texttt{b} & , 23452467) \\
\vdash & (\varepsilon, \varepsilon & , 23452467)
\end{array}
$$

The Nondeterministic Top-Down Automaton III

Theorem 5.15 (Correctness of NTA(G))

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$ *and* NTA(G) *as before. Then, for every* $w \in \Sigma^*$ *and* $z \in [p]^*$,

$$(w, S, \varepsilon) \vdash^* (\varepsilon, \varepsilon, z) \quad \text{iff} \quad z \text{ is a leftmost analysis of } w$$

The Nondeterministic Top-Down Automaton III

Theorem 5.15 (Correctness of NTA(G))

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Proof.

$\Longrightarrow$ (soundness): see exercises

$\Longleftarrow$ (completeness): on the board

□