Compiler Construction

Lecture 12: Syntactic Analysis VIII ($LALR(1)$ Parsing)

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Outline

1. Repetition: $[S]LR(1)$ Parsing
2. $LALR(1)$ Parsing
3. Bottom-Up Parsing of Ambiguous Grammars

The $SLR(1)$ Action Function

Definition ($SLR(1)$ action function)

The $SLR(1)$ action function

$$\mathrm{act} : LR(0)(G) \times \Sigma_\varepsilon \to \{\mathsf{red}\ i \mid i \in [p]\} \cup \{\mathsf{shift}, \mathsf{accept}, \mathsf{error}\}$$

is defined by

$$\mathrm{act}(I, x) := \begin{cases} \mathsf{red}\ i & \text{if } \pi(i) = A \to \alpha, [A \to \alpha\cdot] \in I \ (i \neq 0), \\ & \text{and } x \in \mathrm{fo}(A) \\ \mathsf{shift} & \text{if } [A \to \alpha_1 \cdot x\alpha_2] \in I \text{ and } x \in \Sigma \\ \mathsf{accept} & \text{if } [S' \to S\cdot] \in I \text{ and } x = \varepsilon \\ \mathsf{error} & \text{otherwise} \end{cases}$$

Definition ($SLR(1)$ grammar)

A grammar $G \in CFG_\Sigma$ has the $SLR(1)$ property (notation: $G \in SLR(1)$) if its $SLR(1)$ action function is well defined.

Together, act and the $LR(0)$ goto function (cf. Definition 10.1) form the $SLR(1)$ parsing table of G.

$LR(1)$ Items and Sets I

Observation: not every element of $\mathrm{fo}(A)$ can follow every occurrence of A

$\Longrightarrow$ refinement of $LR(0)$ items by adding possible lookahead symbols

Definition ($LR(1)$ items and sets)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$ be start separated by $S' \to S$.

- If $S' \Rightarrow_r^* \alpha A a w \Rightarrow_r \alpha\beta_1\beta_2 a w$, then $[A \to \beta_1 \cdot \beta_2, a]$ is called an $LR(1)$ item for $\alpha\beta_1$.
- If $S' \Rightarrow_r^* \alpha A \Rightarrow_r \alpha\beta_1\beta_2$, then $[A \to \beta_1 \cdot \beta_2, \varepsilon]$ is called an $LR(1)$ item for $\alpha\beta_1$.
- Given $\gamma \in X^*$, $LR(1)(\gamma)$ denotes the set of all $LR(1)$ items for γ, called the $LR(1)$ set (or: $LR(1)$ information) of γ.
- $LR(1)(G) := \{LR(1)(\gamma) \mid \gamma \in X^*\}$.

$LR(0)$ vs. $LR(1)$

Example

$$G_{LR}: \quad S' \to S \qquad S \to L{=}R \mid R$$
$$L \to *R \mid a \qquad R \to L$$

$LR(0)(G_{LR})$:

$$\begin{array}{lll}
I_0(\varepsilon): & [S' \to \cdot S] & [S \to \cdot L{=}R] \\
 & [S \to \cdot R] & [L \to \cdot *R] \\
 & [L \to \cdot a] & [R \to \cdot L] \\
I_1(S): & [S' \to S\cdot] & \\
I_2(L): & [S \to L \cdot {=}R] & [R \to L\cdot] \\
I_3(R): & [S \to R\cdot] & \\
I_4(*): & [L \to * \cdot R] & [R \to \cdot L] \\
 & [L \to \cdot *R] & [L \to \cdot a] \\
I_5(a): & [L \to a\cdot] & \\
I_6(L{=}): & [S \to L{=} \cdot R] & [R \to \cdot L] \\
 & [L \to \cdot *R] & [L \to \cdot a] \\
I_7(*R): & [L \to *R\cdot] & \\
I_8(*L): & [R \to L\cdot] & \\
I_9(L{=}R): & [S \to L{=}R\cdot] &
\end{array}$$

$LR(1)(G_{LR})$:

$$\begin{array}{lll}
I'_0(\varepsilon): & [S' \to \cdot S, \varepsilon] & [S \to \cdot L{=}R, \varepsilon] \\
 & [S \to \cdot R, \varepsilon] & [L \to \cdot *R, {=}] \\
 & [L \to \cdot a, {=}] & [R \to \cdot L, \varepsilon] \\
 & [L \to \cdot *R, \varepsilon] & [L \to \cdot a, \varepsilon] \\
I'_1(S): & [S' \to S\cdot, \varepsilon] & \\
I'_2(L): & [S \to L \cdot {=}R, \varepsilon] & [R \to L\cdot, \varepsilon] \\
I'_3(R): & [S \to R\cdot, \varepsilon] & \\
I'_4(*): & [L \to * \cdot R, {=}] & [L \to * \cdot R, \varepsilon] \\
 & [R \to \cdot L, {=}] & [R \to \cdot L, \varepsilon] \\
 & [L \to \cdot *R, {=}] & [L \to \cdot a, {=}] \\
 & [L \to \cdot *R, \varepsilon] & [L \to \cdot a, \varepsilon] \\
I'_5(a): & [L \to a\cdot, {=}] & [L \to a\cdot, \varepsilon] \\
I'_6(L{=}): & [S \to L{=} \cdot R, \varepsilon] & [R \to \cdot L, \varepsilon] \\
 & [L \to \cdot *R, \varepsilon] & [L \to \cdot a, \varepsilon] \\
I'_7(*R): & [L \to *R\cdot, {=}] & [L \to *R\cdot, \varepsilon] \\
I'_8(*L): & [R \to L\cdot, {=}] & [R \to L\cdot, \varepsilon] \\
I'_9(L{=}R): & [S \to L{=}R\cdot, \varepsilon] & \\
I'_{10}(L{=}L): & [R \to L\cdot, \varepsilon] & \\
I'_{11}(L{=}*): & [L \to * \cdot R, \varepsilon] & [R \to \cdot L, \varepsilon] \\
 & [L \to \cdot *R, \varepsilon] & [L \to \cdot a, \varepsilon] \\
I'_{12}(L{=}a): & [L \to a\cdot, \varepsilon] & \\
I'_{13}(L{=}*R): & [L \to *R\cdot, \varepsilon] &
\end{array}$$

The $LR(1)$ Action Function

Definition ($LR(1)$ action function)

The $LR(1)$ action function

$$\text{act} : LR(1)(G) \times \Sigma_\varepsilon \to \{\mathsf{red}\ i \mid i \in [p]\} \cup \{\mathsf{shift}, \mathsf{accept}, \mathsf{error}\}$$

is defined by

$$\text{act}(I, x) := \begin{cases} \mathsf{red}\ i & \text{if } \pi(i) = A \to \alpha \text{ and } [A \to \alpha\cdot, x] \in I \\ \mathsf{shift} & \text{if } [A \to \alpha_1 \cdot x\alpha_2, y] \in I \text{ and } x \in \Sigma \\ \mathsf{accept} & \text{if } [S' \to S\cdot, \varepsilon] \in I \text{ and } x = \varepsilon \\ \mathsf{error} & \text{otherwise} \end{cases}$$

Corollary

For every $G \in CFG_\Sigma$, $G \in LR(1)$ iff its $LR(1)$ action function is well defined.

Outline

1. Repetition: $[S]LR(1)$ Parsing
2. $LALR(1)$ Parsing
3. Bottom-Up Parsing of Ambiguous Grammars

- **Motivation:** resolving conflicts using $LR(1)$ too expensive
- Example 11.7/11.13: $|LR(0)(G_{LR})| = 11$, $|LR(1)(G_{LR})| = 15$
- A. Johnstone, E. Scott: *Generalised Reduction Modified LR Parsing for Domain Specific Language Prototyping*, HICSS '02, IEEE, 2002, `http://doi.ieeecomputersociety.org/10.1109/HICSS.2002.994495`:

Grammar	$\|LR(0)(G)\|$	$\|LR(1)(G)\|$
Ansi-C	381	1788
Pascal	368	1395

- **Observation:** potential redundancy by containment of $LR(0)$ sets in $LR(1)$ sets (cf. Corollary 11.9)

Definition 12.1 ($LR(0)$ equivalence)

Let $\mathrm{lr}_0 : LR(1)(G) \to LR(0)(G)$ be defined by

$$\mathrm{lr}_0(I) := \{[A \to \beta_1 \cdot \beta_2] \mid [A \to \beta_1 \cdot \beta_2, x] \in I\}.$$

Two sets $I_1, I_2 \in LR(1)(G)$ are called $LR(0)$-equivalent (notation: $I_1 \sim_0 I_2$) if $\mathrm{lr}_0(I_1) = \mathrm{lr}_0(I_2)$.

$LR(0)$ Equivalence

Example 12.2 (cf. Example 11.7/11.13)

G_{LR} : $S' \to S$ $\quad S \to L{=}R \mid R$
$L \to {*}R \mid \mathtt{a}$ $\quad R \to L$

$LR(0)(G_{LR})$:

$I_0(\varepsilon)$:	$[S' \to \cdot S]$	$[S \to \cdot L{=}R]$
	$[S \to \cdot R]$	$[L \to \cdot {*}R]$
	$[L \to \cdot \mathtt{a}]$	$[R \to \cdot L]$
$I_1(S)$:	$[S' \to S \cdot]$	
$I_2(L)$:	$[S \to L \cdot {=}R]$	$[R \to L \cdot]$
$I_3(R)$:	$[S \to R \cdot]$	
$I_4({*})$:	$[L \to {*} \cdot R]$	$[R \to \cdot L]$
	$[L \to \cdot {*}R]$	$[L \to \cdot \mathtt{a}]$
$I_5(\mathtt{a})$:	$[L \to \mathtt{a} \cdot]$	
$I_6(L{=})$:	$[S \to L{=} \cdot R]$	$[R \to \cdot L]$
	$[L \to \cdot {*}R]$	$[L \to \cdot \mathtt{a}]$
$I_7({*}R)$:	$[L \to {*}R \cdot]$	
$I_8({*}L)$:	$[R \to L \cdot]$	
$I_9(L{=}R)$:	$[S \to L{=}R \cdot]$	

$$\Longrightarrow \quad \begin{array}{ccc} I'_4 & \sim_0 & I'_{11} \\ I'_5 & \sim_0 & I'_{12} \\ I'_7 & \sim_0 & I'_{13} \\ I'_8 & \sim_0 & I'_{10} \end{array}$$

$LR(1)(G_{LR})$:

$I'_0(\varepsilon)$:	$[S' \to \cdot S, \varepsilon]$	$[S \to \cdot L{=}R, \varepsilon]$
	$[S \to \cdot R, \varepsilon]$	$[L \to \cdot {*}R, {=}]$
	$[L \to \cdot \mathtt{a}, {=}]$	$[R \to \cdot L, \varepsilon]$
	$[L \to \cdot {*}R, \varepsilon]$	$[L \to \cdot \mathtt{a}, \varepsilon]$
$I'_1(S)$:	$[S' \to S \cdot, \varepsilon]$	
$I'_2(L)$:	$[S \to L \cdot {=}R, \varepsilon]$	$[R \to L \cdot, \varepsilon]$
$I'_3(R)$:	$[S \to R \cdot, \varepsilon]$	
$I'_4({*})$:	$[L \to {*} \cdot R, {=}]$	$[L \to {*} \cdot R, \varepsilon]$
	$[R \to \cdot L, {=}]$	$[R \to \cdot L, \varepsilon]$
	$[L \to \cdot {*}R, {=}]$	$[L \to \cdot \mathtt{a}, {=}]$
	$[L \to \cdot {*}R, \varepsilon]$	$[L \to \cdot \mathtt{a}, \varepsilon]$
$I'_5(\mathtt{a})$:	$[L \to \mathtt{a} \cdot, {=}]$	$[L \to \mathtt{a} \cdot, \varepsilon]$
$I'_6(L{=})$:	$[S \to L{=} \cdot R, \varepsilon]$	$[R \to \cdot L, \varepsilon]$
	$[L \to \cdot {*}R, \varepsilon]$	$[L \to \cdot \mathtt{a}, \varepsilon]$
$I'_7({*}R)$:	$[L \to {*}R \cdot, {=}]$	$[L \to {*}R \cdot, \varepsilon]$
$I'_8({*}L)$:	$[R \to L \cdot, {=}]$	$[R \to L \cdot, \varepsilon]$
$I'_9(L{=}R)$:	$[S \to L{=}R \cdot, \varepsilon]$	
$I'_{10}(L{=}L)$:	$[R \to L \cdot, \varepsilon]$	
$I'_{11}(L{=}{*})$:	$[L \to {*} \cdot R, \varepsilon]$	$[R \to \cdot L, \varepsilon]$
	$[L \to \cdot {*}R, \varepsilon]$	$[L \to \cdot \mathtt{a}, \varepsilon]$
$I'_{12}(L{=}\mathtt{a})$:	$[L \to \mathtt{a} \cdot, \varepsilon]$	
$I'_{13}(L{=}{*}R)$:	$[L \to {*}R \cdot, \varepsilon]$	

$LALR(1)$ Sets I

Corollary 12.3

For every $G \in CFG_\Sigma$, $|LR(1)(G)/\sim_0| = |LR(0)(G)|$.

Idea: merge $LR(0)$-equivalent $LR(1)$ sets (maintaining the lookahead information, but possibly introducing conflicts)

Definition 12.4 ($LALR(1)$ sets)

Let $G \in CFG_\Sigma$.

- An information $I \in LR(1)(G)$ determines the $LALR(1)$ set
$$\bigcup[I]_{\sim_0} = \bigcup\{I' \in LR(1)(G) \mid I' \sim_0 I\}.$$
- The set of all $LALR(1)$ sets of G is denoted by $LALR(1)(G)$.

Remark: by Corollary 12.3, $|LALR(1)(G)| = |LR(0)(G)|$
(but $LALR(1)$ sets provide additional lookahead information)

Example 12.5 (cf. Example 12.2)

$G_{LR}: S' \to S \quad S \to L{=}R \mid R \quad L \to {*}R \mid \mathtt{a} \quad R \to L$

$LR(0)(G_{LR})$:

$I_0(\varepsilon)$:	$[S' \to \cdot S]$	$[S \to \cdot L{=}R]$
	$[S \to \cdot R]$	$[L \to \cdot {*}R]$
	$[L \to \cdot \mathtt{a}]$	$[R \to \cdot L]$
$I_1(S)$:	$[S' \to S \cdot]$	
$I_2(L)$:	$[S \to L \cdot {=}R]$	$[R \to L \cdot]$
$I_3(R)$:	$[S \to R \cdot]$	
$I_4({*})$:	$[L \to {*} \cdot R]$	$[R \to \cdot L]$
	$[L \to \cdot {*}R]$	$[L \to \cdot \mathtt{a}]$
$I_5(\mathtt{a})$:	$[L \to \mathtt{a} \cdot]$	
$I_6(L{=})$:	$[S \to L{=} \cdot R]$	$[R \to \cdot L]$
	$[L \to \cdot {*}R]$	$[L \to \cdot \mathtt{a}]$
$I_7({*}R)$:	$[L \to {*}R \cdot]$	
$I_8({*}L)$:	$[R \to L \cdot]$	
$I_9(L{=}R)$:	$[S \to L{=}R \cdot]$	

$LALR(1)(G_{LR})$:

$I''_0 := I'_0$:	$[S' \to \cdot S, \varepsilon]$	$[S \to \cdot L{=}R, \varepsilon]$
	$[S \to \cdot R, \varepsilon]$	$[L \to \cdot {*}R, {=}/\varepsilon]$
	$[L \to \cdot \mathtt{a}, {=}/\varepsilon]$	$[R \to \cdot L, \varepsilon]$
$I''_1 := I'_1$:	$[S' \to S \cdot, \varepsilon]$	
$I''_2 := I'_2$:	$[S \to L \cdot {=}R, \varepsilon]$	$[R \to L \cdot, \varepsilon]$
$I''_3 := I'_3$:	$[S \to R \cdot, \varepsilon]$	
$I''_4 := I'_4 \cup I'_{11}$:	$[L \to {*} \cdot R, {=}/\varepsilon]$	$[R \to \cdot L, {=}/\varepsilon]$
	$[L \to \cdot {*}R, {=}/\varepsilon]$	$[L \to \cdot \mathtt{a}, {=}/\varepsilon]$
$I''_5 := I'_5 \cup I'_{12}$:	$[L \to \mathtt{a} \cdot, {=}/\varepsilon]$	
$I''_6 := I'_6$:	$[S \to L{=} \cdot R, \varepsilon]$	$[R \to \cdot L, \varepsilon]$
	$[L \to \cdot {*}R, \varepsilon]$	$[L \to \cdot \mathtt{a}, \varepsilon]$
$I''_7 := I'_7 \cup I'_{13}$:	$[L \to {*}R \cdot, {=}/\varepsilon]$	
$I''_8 := I'_8 \cup I'_{10}$:	$[R \to L \cdot, {=}/\varepsilon]$	
$I''_9 := I'_9$:	$[S \to L{=}R \cdot, \varepsilon]$	

The $LALR(1)$ Action Function

The $LALR(1)$ action function is defined in analogy to the $LR(1)$ case (Definition 11.14).

Definition 12.6 ($LALR(1)$ action function)

The $LALR(1)$ action function

$$\mathrm{act} : LALR(1)(G) \times \Sigma_\varepsilon \to \{\mathsf{red}\ i \mid i \in [p]\} \cup \{\mathsf{shift}, \mathsf{accept}, \mathsf{error}\}$$

is defined by

$$\mathrm{act}(I, x) := \begin{cases} \mathsf{red}\ i & \text{if } \pi(i) = A \to \alpha \text{ and } [A \to \alpha\cdot, x] \in I \\ \mathsf{shift} & \text{if } [A \to \alpha_1 \cdot x\alpha_2, y] \in I \text{ and } x \in \Sigma \\ \mathsf{accept} & \text{if } [S' \to S\cdot, \varepsilon] \in I \text{ and } x = \varepsilon \\ \mathsf{error} & \text{otherwise} \end{cases}$$

Definition 12.7 ($LALR(1)$ grammar)

A grammar $G \in CFG_\Sigma$ has the $LALR(1)$ property (notation: $G \in LALR(1)$) if its $LALR(1)$ action function is well defined.

The $LALR(1)$ goto Function

Example 12.8 (cf. Example 12.5)

$G_{LR} \in LALR(1)$

Also the $LR(1)$ goto function (Definition 11.16) carries over to the $LALR(1)$ case. Reason:

Lemma 12.9

Let $G \in CFG_\Sigma$ and $I_1, I_2 \in LR(1)(G)$ such that $I_1 \sim_0 I_2$. Then, for every $Y \in X$, $\text{goto}(I_1, Y) \sim_0 \text{goto}(I_2, Y)$.

Again, act and goto form the $LALR(1)$ parsing table of G.

Example 12.10 (cf. Example 12.5)

$LALR(1)(G_{LR})$	act/goto$\|_{\Sigma_\varepsilon}$				goto$\|_N$		
	`*`	`=`	`a`	ε	S	L	R
I''_0	shift/I''_4		shift/I''_5		I''_1	I''_2	I''_3
I''_1				accept			
I''_2		shift/I''_6		red 5			
I''_3				red 2			
I''_4	shift/I''_4		shift/I''_5			I''_8	I''_7
I''_5		red 4		red 4			
I''_6	shift/I''_4		shift/I''_5			I''_8	I''_9
I''_7		red 3		red 3			
I''_8		red 5		red 5			
I''_9				red 1			

(empty = error/$\emptyset$)

$LALR(1)$ Conflicts

But: merging of $LR(1)$ sets can produce new conflicts (also see exercises):

Example 12.11

$G: S' \to S \quad S \to \mathtt{a}A\mathtt{d} \mid \mathtt{b}B\mathtt{d} \mid \mathtt{a}B\mathtt{e} \mid \mathtt{b}A\mathtt{e} \quad A \to \mathtt{c} \quad B \to \mathtt{c}$

$LR(1)(\varepsilon)$: $[S' \to \cdot S, \varepsilon]$ $[S \to \cdot \mathtt{a}A\mathtt{d}, \varepsilon]$ $[S \to \cdot \mathtt{b}B\mathtt{d}, \varepsilon]$ $[S \to \cdot \mathtt{a}B\mathtt{e}, \varepsilon]$ $[S \to \cdot \mathtt{b}A\mathtt{e}, \varepsilon]$

$LR(1)(S)$: $[S' \to S \cdot, \varepsilon]$

$LR(1)(\mathtt{a})$: $[S \to \mathtt{a} \cdot A\mathtt{d}, \varepsilon]$ $[S \to \mathtt{a} \cdot B\mathtt{e}, \varepsilon]$ $[A \to \cdot \mathtt{c}, \mathtt{d}]$ $[B \to \cdot \mathtt{c}, \mathtt{e}]$

$LR(1)(\mathtt{b})$: $[S \to \mathtt{b} \cdot B\mathtt{d}, \varepsilon]$ $[S \to \mathtt{b} \cdot A\mathtt{e}, \varepsilon]$ $[B \to \cdot \mathtt{c}, \mathtt{d}]$ $[A \to \cdot \mathtt{c}, \mathtt{e}]$

$LR(1)(\mathtt{a}A)$: $[S \to \mathtt{a}A \cdot \mathtt{d}, \varepsilon]$ $LR(1)(\mathtt{a}B)$: $[S \to \mathtt{a}B \cdot \mathtt{e}, \varepsilon]$

$LR(1)(\mathtt{ac})$: $[A \to \mathtt{c} \cdot, \mathtt{d}]$ $[B \to \mathtt{c} \cdot, \mathtt{e}]$

$LR(1)(\mathtt{b}B)$: $[S \to \mathtt{b}B \cdot \mathtt{d}, \varepsilon]$ $LR(1)(\mathtt{b}A)$: $[S \to \mathtt{b}A \cdot \mathtt{e}, \varepsilon]$

$LR(1)(\mathtt{bc})$: $[B \to \mathtt{c} \cdot, \mathtt{d}]$ $[A \to \mathtt{c} \cdot, \mathtt{e}]$

$LR(1)(\mathtt{a}A\mathtt{d})$: $[S \to \mathtt{a}A\mathtt{d} \cdot, \varepsilon]$ $LR(1)(\mathtt{a}B\mathtt{e})$: $[S \to \mathtt{a}B\mathtt{e} \cdot, \varepsilon]$

$LR(1)(\mathtt{b}B\mathtt{d})$: $[S \to \mathtt{b}B\mathtt{d} \cdot, \varepsilon]$ $LR(1)(\mathtt{b}A\mathtt{e})$: $[S \to \mathtt{b}A\mathtt{e} \cdot, \varepsilon]$

no conflicts $\implies G \in LR(1)$

$LR(1)(\mathtt{ac}) \sim_0 LR(1)(\mathtt{bc})$, but $LR(1)(\mathtt{ac}) \cup LR(1)(\mathtt{bc})$ has conflicts
$\implies G \notin LALR(1)$

Efficient Construction of $LALR(1)$ Parsers

Naive algorithm to construct $LALR(1)$ parser for $G \in CFG_\Sigma$:

1. Construct $LR(1)(G)$
2. Determine and merge $LR(0)$-equivalent $LR(1)$ sets

Problem: no reduction of peak space requirement

Idea of improved algorithm (see Aho/Lam/Sethi/Ullman: *Compilers: Principles, Techniques, and Tools*, 2nd ed., p. 270ff):

1. Represent each set of items by its kernel, i.e., by the items of the form $[S' \to \cdot S, \varepsilon]$ or $[A \to \beta_1 \cdot \beta_2, x]$ where $\beta_1 \neq \varepsilon$
2. Construct $LALR(1)$ kernels from $LR(0)$ kernels similarly to $LR(1)$ items
3. Compute $LALR(1)$ sets by taking the ε-closure

(applied in `yacc` parser generator)

Outline

1. Repetition: $[S]LR(1)$ Parsing

2. $LALR(1)$ Parsing

3. Bottom-Up Parsing of Ambiguous Grammars

Ambiguous Grammars

Reminder (Definition 5.6): a context-free grammar $G \in CFG_\Sigma$ is called unambiguous if every word $w \in L(G)$ has exactly one syntax tree. Otherwise it is called ambiguous.

Lemma 12.12

If $G \in CFG_\Sigma$ is ambiguous, then $G \notin \bigcup_{k \in \mathbb{N}} LR(k)$.

Proof.

Assume that there exist $k \in \mathbb{N}$ and $G \in LR(k)$ such that G is ambiguous. Hence there exists $w \in L(G)$ with different right derivations. Let $\alpha A v$ be the last common sentence of the two derivations (i.e., $\beta \neq \beta'$):

$$S \Rightarrow_r^* \alpha A v \begin{cases} \Rightarrow_r \ \alpha \beta v \ \Rightarrow_r^* \ w \\ \Rightarrow_r \ \alpha \beta' v \ \Rightarrow_r^* \ w \end{cases}$$

But since $\text{first}_k(v) = \text{first}_k(v)$ for every $v \in \Sigma^*$, Definition 9.7 yields that $\beta = \beta'$. Contradiction □

However ambiguity is a natural specification method which generally avoids involved syntactic constructs.

Example 12.13 (Simple arithmetic expressions)

$G: E' \to E \quad E \to E{+}E \mid E{*}E \mid \texttt{a}$

Precedence: $* > +$ Associativity: left

(thus: a+a*a+a := (a+(a*a))+a)

$LR(0)(G)$:

$I_0 := LR(0)(\varepsilon)$: $[E' \to \cdot E]$ $[E \to \cdot E{+}E]$ $[E \to \cdot E{*}E]$ $[E \to \cdot \texttt{a}]$
$I_1 := LR(0)(E)$: $[E' \to E\cdot]$ $[E \to E \cdot {+}E]$ $[E \to E \cdot {*}E]$
$I_2 := LR(0)(\texttt{a})$: $[E \to \texttt{a}\cdot]$
$I_3 := LR(0)(E{+})$: $[E \to E{+} \cdot E]$ $[E \to \cdot E{+}E]$ $[E \to \cdot E{*}E]$ $[E \to \cdot \texttt{a}]$
$I_4 := LR(0)(E{*})$: $[E \to E{*} \cdot E]$ $[E \to \cdot E{+}E]$ $[E \to \cdot E{*}E]$ $[E \to \cdot \texttt{a}]$
$I_5 := LR(0)(E{+}E)$: $[E \to E{+}E\cdot]$ $[E \to E \cdot {+}E]$ $[E \to E \cdot {*}E]$
$I_6 := LR(0)(E{*}E)$: $[E \to E{*}E\cdot]$ $[E \to E \cdot {+}E]$ $[E \to E \cdot {*}E]$

Conflicts: I_1: $SLR(1)$-solvable (reduce on ε, shift on $+/*$)
I_5, I_6: not $SLR(1)$-solvable ($+, * \in \mathrm{fo}(E)$)

Solution:

I_5: $* > + \implies \mathrm{act}(I_5, *) :=$ shift, $+$ left assoc. $\implies \mathrm{act}(I_5, +) :=$ red 1
I_6: $* > + \implies \mathrm{act}(I_6, +) :=$ red 2, $*$ left assoc. $\implies \mathrm{act}(I_6, *) :=$ red 2

Example 12.14 (“Dangling `else`”)

$G : S' \to S \quad S \to \mathtt{i}S\mathtt{e}S \mid \mathtt{i}S \mid \mathtt{a}$

Ambiguity: `iiaea` := (1) `i(iaea)` (common) or (2) `i(ia)ea`

$LR(0)(G)$:

$I_0 := LR(0)(\varepsilon) : \quad [S' \to \cdot S] \quad [S \to \cdot \mathtt{i}S\mathtt{e}S] \; [S \to \cdot \mathtt{i}S] \; [S \to \cdot \mathtt{a}]$

$I_1 := LR(0)(S) : \quad [S' \to S \cdot]$

$I_2 := LR(0)(\mathtt{i}) : \quad [S \to \mathtt{i} \cdot S\mathtt{e}S] \; [S \to \mathtt{i} \cdot S] \quad [S \to \cdot \mathtt{i}S\mathtt{e}S] \; [S \to \cdot \mathtt{i}S] \quad [S \to \cdot \mathtt{a}]$

$I_3 := LR(0)(\mathtt{a}) : \quad [S \to \mathtt{a} \cdot]$

$I_4 := LR(0)(\mathtt{i}S) : \quad [S \to \mathtt{i}S \cdot \mathtt{e}S] \; [S \to \mathtt{i}S \cdot]$

$I_5 := LR(0)(\mathtt{i}S\mathtt{e}) : \quad [S \to \mathtt{i}S\mathtt{e} \cdot S] \; [S \to \cdot \mathtt{i}S\mathtt{e}S] \; [S \to \cdot \mathtt{i}S] \; [S \to \cdot \mathtt{a}]$

$I_6 := LR(0)(\mathtt{i}S\mathtt{e}S) : [S \to \mathtt{i}S\mathtt{e}S \cdot]$

Conflict in I_4: $\mathtt{e} \in \mathrm{fo}(S) \implies$ not $SLR(1)$-solvable

Solution (1): $\mathrm{act}(I_4, \mathtt{e}) :=$ shift