Compiler Construction

Lecture 3: Lexical Analysis II (The Matching Problem)

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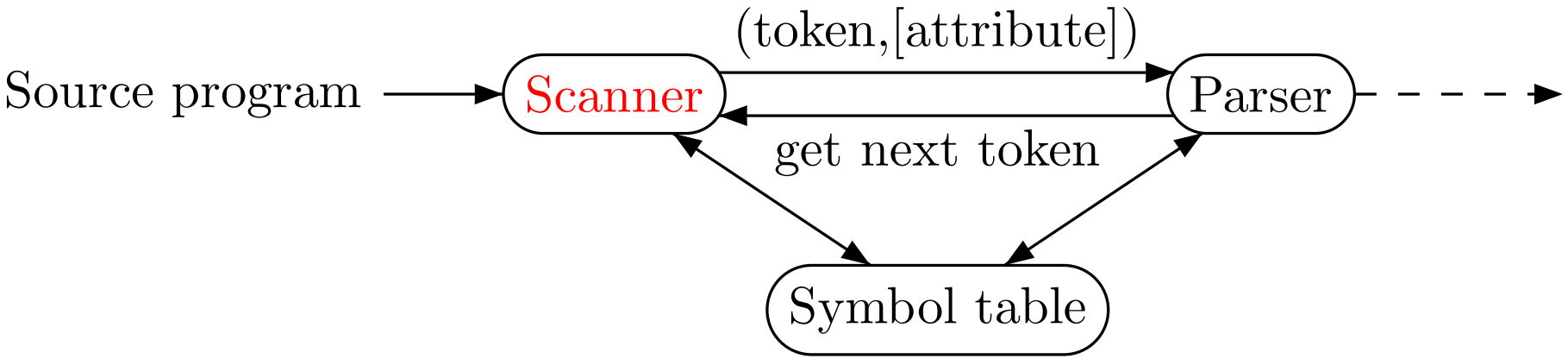
Outline

Lexical Analysis

Definition

The goal of lexical analysis is to decompose a source program into a sequence of lexemes and their transformation into a sequence of symbols.

The corresponding program is called a scanner:



Example:

$$\ldots\,\texttt{␣x1␣:=y2+␣1␣;␣}\ldots$$

$$\Downarrow$$

$$\ldots(\mathsf{id}, p_1)(\mathsf{gets},)(\mathsf{id}, p_2)(\mathsf{plus},)(\mathsf{int}, 1)(\mathsf{sem},)\ldots$$

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The Simple Matching Problem I

Problem 3.1 (Simple matching problem)

Given $\alpha \in RE_\Omega$ and $w \in \Omega^*$, decide whether $w \in [\![\alpha]\!]$ or not.

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This problem can be solved using the following concept:

Definition 3.2 (Finite automaton)

A nondeterministic finite automaton (NFA) is of the form $\mathfrak{A} = \langle Q, \Omega, \delta, q_0, F \rangle$ where

- Q is a finite set of states
- Ω denotes the input alphabet
- $\delta : Q \times \Omega_\varepsilon \to 2^Q$ is the transition function where $\Omega_\varepsilon := \Omega \cup \{\varepsilon\}$ (notation: $q \xrightarrow{x} q'$ for $q' \in \delta(q, x)$)
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of final states

The set of all NFA over Ω is denoted by NFA_Ω.
If $\delta(q, \varepsilon) = \emptyset$ and $|\delta(q, a)| = 1$ for every $q \in Q$ and $a \in \Omega$ (i.e., $\delta : Q \times \Omega \to Q$), then $\mathfrak{A}$ is called deterministic (DFA). Notation: DFA_Ω

The Simple Matching Problem II

Definition 3.3 (Acceptance condition)

Let $\mathfrak{A} = \langle Q, \Omega, \delta, q_0, F \rangle \in NFA_\Omega$ and $w = a_1 \dots a_n \in \Omega^*$.

- A w-labeled $\mathfrak{A}$-run from q_1 to q_2 is a sequence of transitions

$$q_1 \xrightarrow{\varepsilon}{}^* \xrightarrow{a_1} \xrightarrow{\varepsilon}{}^* \xrightarrow{a_2} \xrightarrow{\varepsilon}{}^* \dots \xrightarrow{\varepsilon}{}^* \xrightarrow{a_n} \xrightarrow{\varepsilon}{}^* q_2$$

- $\mathfrak{A}$ accepts w if there is a w-labeled $\mathfrak{A}$-run from q_0 to some $q \in F$
- The language recognized by $\mathfrak{A}$ is

$$L(\mathfrak{A}) := \{w \in \Omega^* \mid \mathfrak{A} \text{ accepts } w\}$$

- A language $L \subseteq \Omega^*$ is called NFA-recognizable if there exists a NFA $\mathfrak{A}$ such that $L(\mathfrak{A}) = L$

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Example 3.4

NFA for $a^*b \mid a^*$ (on the board)

The Simple Matching Problem III

Remarks:

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- For $\mathfrak{A} \in DFA_\Omega$, the acceptance condition yields $\widehat{\delta} : Q \times \Omega^* \to Q$ with $\widehat{\delta}(q, \varepsilon) = q$ and $\widehat{\delta}(q, aw) = \widehat{\delta}(\delta(q, a), w)$, and

$$L(\mathfrak{A}) = \{w \in \Omega^* \mid \widehat{\delta}(q_0, w) \in F\}.$$

The DFA Method I

Known from *Automata Theory and Formal Languages*:

Algorithm 3.5 (DFA method)

Input: *regular expression* $\alpha \in RE_\Omega$, *input string* $w \in \Omega^*$

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Procedure:

1. *using Kleene's Theorem, construct* $\mathfrak{A}_\alpha \in NFA_\Omega$ *such that* $L(\mathfrak{A}_\alpha) = [\![\alpha]\!]$
2. *apply powerset construction to obtain* $\mathfrak{A}'_\alpha = \langle Q', \Omega, \delta', q'_0, F' \rangle \in DFA_\Omega$ *with* $L(\mathfrak{A}'_\alpha) = L(\mathfrak{A}_\alpha) = [\![\alpha]\!]$
3. *solve the matching problem by deciding whether* $\widehat{\delta'}(q'_0, w) \in F'$

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Output: *"yes" or "no"*

The DFA Method II

The powerset construction involves the following concept:

Definition 3.6 (ε-closure)

Let $\mathfrak{A} = \langle Q, \Omega, \delta, q_0, F \rangle \in NFA_\Omega$. The ε-closure $\varepsilon(T) \subseteq Q$ of a subset $T \subseteq Q$ is defined by

- $T \subseteq \varepsilon(T)$ and
- if $q \in \varepsilon(T)$, then $\delta(q, \varepsilon) \subseteq \varepsilon(T)$

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Example 3.7

1. Kleene's Theorem (in general)
2. Powerset construction
   (for NFA $\mathfrak{A}_\alpha$ with $\alpha := a^*b \mid a^*$; cf. Example 3.4)

(on the board)

Complexity of DFA Method

1. In construction phase:
   - Kleene method: time and space $\mathcal{O}(|\alpha|)$ ($|\alpha|$:= length of α)
   - Powerset construction: time and space $\mathcal{O}(2^{|\mathfrak{A}_\alpha|}) = \mathcal{O}(2^{|\alpha|})$ ($|\mathfrak{A}_\alpha|$:= # of states)

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$\Longrightarrow$ Nice runtime behavior but memory requirements too high (and exponential time in construction phase)

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Variables: $T \subseteq Q$, $a \in \Omega$

Procedure:
$T := \varepsilon(\{q_0\})$;
while $w \neq \varepsilon$ **do**
  $a := \mathbf{head}(w)$;
  $T := \varepsilon\left(\bigcup_{q \in T} \delta(q, a)\right)$;
  $w := \mathbf{tail}(w)$
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Example 3.9 (cf. Example 3.4)

NFA $\mathfrak{A}_\alpha$ for $\alpha := a^*b \mid a^*$, $w := aab$

Complexity Analysis

For NFA method at runtime:

- Space: $\mathcal{O}(|\alpha|)$ (for storing NFA and T)
- Time: $\mathcal{O}(|\alpha| \cdot |w|)$
  (In the loop's body, $|T|$ states need to be considered. Here evaluation of $\delta(q, a)$ is considered as *one* operation.)

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Comparison:

Method	Space	Time (for "$w \in [\![\alpha]\!]$?")
DFA	$\mathcal{O}(2^{\|\alpha\|})$	$\mathcal{O}(\|w\|)$
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In practice:

- Exponential blowup of DFA method usually does not occur in "real" applications ($\Longrightarrow$ used in `[f]lex`)
- Improvement of NFA method: caching of transitions $\widehat{\delta}(T, a)$
  $\Longrightarrow$ combination of both methods

Outline

The Extended Matching Problem I

Definition 3.10

Let $n \geq 1$ and $\alpha_1, \ldots, \alpha_n \in RE_\Omega$ with $\varepsilon \notin [\![\alpha_i]\!] \neq \emptyset$ for every $i \in [n]$ $(= \{1, \ldots, n\})$. Let $\Sigma := \{T_1, \ldots, T_n\}$ be an alphabet of corresponding tokens and $w \in \Omega^+$. If $w_1, \ldots, w_k \in \Omega^+$ such that

- $w = w_1 \ldots w_k$ and
- for every $j \in [k]$ there exists $i_j \in [n]$ such that $w_j \in [\![\alpha_{i_j}]\!]$,

then

- $(w_1, \ldots, w_k)$ is called a decomposition and
- $(T_{i_1}, \ldots, T_{i_k})$ is called an analysis

of w w.r.t. $\alpha_1, \ldots, \alpha_n$.

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of w w.r.t. $\alpha_1, \ldots, \alpha_n$.

Problem 3.11 (Extended matching problem)

Given $\alpha_1, \ldots, \alpha_n \in RE_\Omega$ and $w \in \Omega^+$, decide whether there exists a decomposition of w w.r.t. $\alpha_1, \ldots, \alpha_n$ and determine a corresponding analysis.

The Extended Matching Problem II

Observation: neither the decomposition nor the analysis are uniquely determined

Example 3.12

1. $\alpha = a^+$, $w = aa$
   $\Longrightarrow$ two decompositions (aa) and (a, a) with unique analysis each

The Extended Matching Problem II

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2. $\alpha_1 = a \mid b$, $\alpha_2 = a \mid c$, $w = a$
   $\Longrightarrow$ unique decomposition (a) but two analyses (T_1) and (T_2)

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Goal: make both unique $\Longrightarrow$ deterministic scanning