Compiler Construction

Lecture 9: Syntax Analysis V ($LR(k)$ Grammars)

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Summer Semester 2012

Outline

Approach:

1. Given $G \in CFG_\Sigma$, construct a nondeterministic bottom-up parsing automaton (NBA) which accepts $L(G)$ and which additionally computes corresponding (reversed) rightmost analyses
   - input alphabet: Σ
   - pushdown alphabet: X
   - output alphabet: $[p]$ (where $p := |P|$)
   - state set: omitted
   - transitions:

     shift: shifting input symbols onto the pushdown

     reduce: replacing the right-hand side of a production by its left-hand side (= inverse expansion steps)
2. Remove nondeterminism by allowing lookahead on the input: $G \in LR(k)$ iff $L(G)$ recognizable by deterministic bottom-up parsing automaton with lookahead of k symbols

Nondeterministic Bottom-Up Automaton

Definition (Nondeterministic bottom-up parsing automaton)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$. The nondeterministic bottom-up parsing automaton of G, $\mathrm{NBA}(G)$, is defined by the following components.

- Input alphabet: Σ
- Pushdown alphabet: X
- Output alphabet: $[p]$
- Configurations: $\Sigma^* \times X^* \times [p]^*$ (top of pushdown to the right)
- Transitions for $w \in \Sigma^*$, $\alpha \in X^*$, and $z \in [p]^*$:

  shifting steps: $(aw, \alpha, z) \vdash (w, \alpha a, z)$ if $a \in \Sigma$
  reduction steps: $(w, \alpha\beta, z) \vdash (w, \alpha A, zi)$ if $\pi_i = A \to \beta$
- Initial configuration for $w \in \Sigma^*$: $(w, \varepsilon, \varepsilon)$
- Final configurations: $\{\varepsilon\} \times \{S\} \times [p]^*$

Outline

Theorem 9.1 (Correctness of $\mathrm{NBA}(G)$)

Let $G = \langle N, \Sigma, P, S\rangle \in CFG_\Sigma$ and $\mathrm{NBA}(G)$ as before. Then, for every $w \in \Sigma^*$ and $z \in [p]^*$,

$$(w, \varepsilon, \varepsilon) \vdash^* (\varepsilon, S, z) \quad \textit{iff} \quad \overleftarrow{z} \textit{ is a rightmost analysis of } w$$

Proof.

similar to the top-down case (Theorem 6.1) □

Nondeterminisn in $\mathrm{NBA}(G)$

Observation: $\mathrm{NBA}(G)$ is generally nondeterministic

- Shift or reduce? Example:

$$(bw, \alpha a, z) \vdash \begin{cases} (w, \alpha ab, z) \\ (bw, \alpha A, zi) \end{cases} \text{if } \pi_i = A \to a$$

- If reduce: which "handle" β? Example:

$$(w, \alpha ab, z) \vdash \begin{cases} (w, \alpha A, zi) \\ (w, \alpha aB, zj) \end{cases} \text{if } \pi_i = A \to ab \text{ and } \pi_j = B \to b$$

- If reduce β: which left-hand side A? Example:

$$(w, \alpha a, z) \vdash \begin{cases} (w, \alpha A, zi) \\ (w, \alpha B, zj) \end{cases} \text{if } \pi_i = A \to a \text{ and } \pi_j = B \to a$$

- When to terminate parsing? Example:

$$\underbrace{(\varepsilon, S, z)}_{\text{final}} \vdash (\varepsilon, A, zi) \text{ if } \pi_i = A \to S$$

Resolving Termination Nondeterminism I

General assumption to avoid nondeterminism of last type:
every grammar is start separated

Definition 9.2 (Start separation)

A grammar $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$ is called start separated if S only occurs in productions of the form $S \rightarrow A$ where $A \neq S$.

Remarks:

- Start separation always possible by adding $S' \rightarrow S$ with new start symbol S'
- From now on consider only reduced grammars of this form ($\pi_0 := S' \rightarrow S$)

Resolving Termination Nondeterminism II

Start separation removes last form of nondeterminism ("When to terminate parsing?"):

Lemma 9.3

If $G \in CFG_\Sigma$ is start separated, then no successor of a final configuration (ε, S', z) in $\mathrm{NBA}(G)$ is again a final configuration.
(Thus parsing should be stopped in the first final configuration.)

Proof.

- To (ε, S', z), only reductions by ε-productions can be applied:

$$(\varepsilon, S', z) \vdash (\varepsilon, S'A, zi) \quad \text{if } \pi_i = A \to \varepsilon$$

- Thereafter, only reductions by productions of the form $A_0 \to A_1 \dots A_n$ $(n \geq 0)$ can be applied
- Every resulting configuration is of the (non-final) form

$$(\varepsilon, S'B_1 \dots B_k, z) \quad \text{where } k \geq 1$$

□

Outline

Goal: resolve remaining nondeterminism of $\mathrm{NBA}(G)$ by supporting lookahead of $k \in \mathbb{N}$ symbols on the input

$\Longrightarrow$ $LR(k)$: reading of input from left to right with k-lookahead, computing a rightmost analysis

Definition 9.4 ($LR(k)$ grammar)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$ be start separated and $k \in \mathbb{N}$. Then G has the $LR(k)$ property (notation: $G \in LR(k)$) if for all rightmost derivations of the form

$$S \begin{cases} \Rightarrow_r^* \alpha A w \Rightarrow_r \alpha\beta w \\ \Rightarrow_r^* \gamma B x \Rightarrow_r \alpha\beta y \end{cases}$$

such that $\mathrm{first}_k(w) = \mathrm{first}_k(y)$, it follows that $\alpha = \gamma$, $A = B$, and $x = y$.

Remarks:

- If $G \in LR(k)$, then the reduction of $\alpha\beta w$ to $\alpha A w$ is already determined by $\mathrm{first}_k(w)$.
- Therefore $\mathrm{NBA}(G)$ in configuration $(w, \alpha\beta, z)$ can decide to reduce and how to reduce.
- Computation of $\mathrm{NBA}(G)$ for $S \Rightarrow_r^* \alpha A w \Rightarrow_r \alpha\beta w$:

$$(w'w, \varepsilon, \varepsilon) \vdash^* (w, \alpha\beta, z) \overset{\text{red } i}{\vdash} (w, \alpha A, zi) \vdash \ldots$$

  where $\pi_i = A \to \beta$
- Computation of $\mathrm{NBA}(G)$ for $S \Rightarrow_r^* \gamma B x \Rightarrow_r \alpha\beta y$:
  - with direct reduction ($y = x$, $\alpha\beta = \gamma\delta$, $\pi_j = B \to \delta$):

$$(y'y, \varepsilon, \varepsilon) \vdash^* (y, \alpha\beta, z') = (x, \gamma\delta, z') \overset{\text{red } j}{\vdash} (x, \gamma B, z'j) \vdash \ldots$$

  - with previous shifts ($y = x'x$, $\alpha\beta x' = \gamma\delta$, $\pi_j = B \to \delta$):

$$\begin{aligned}(y'y, \varepsilon, \varepsilon) &\vdash^* (y, \alpha\beta, z') = (x'x, \alpha\beta, z') \\ &\overset{\text{shift}^*}{\vdash} (x, \alpha\beta x', z') = (x, \gamma\delta, z') \\ &\overset{\text{red } j}{\vdash} (x, \gamma B, z'j) \vdash \ldots\end{aligned}$$

Outline

$LR(0)$ Grammars

The case $k = 0$ is relevant (in contrast to $LL(0)$): here the decision is just based on the contents of the pushdown, without any lookahead.

Corollary 9.5 ($LR(0)$ grammar)

$G \in CFG_\Sigma$ has the $LR(0)$ property if for all rightmost derivations of the form

$$S \begin{cases} \Rightarrow_r^* \alpha A w \Rightarrow_r \alpha\beta w \\ \Rightarrow_r^* \gamma B x \Rightarrow_r \alpha\beta y \end{cases}$$

it follows that $\alpha = \gamma$, $A = B$, and $x = y$.

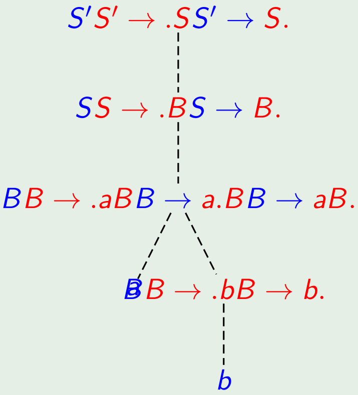
Goal: derive a finite information from the pushdown which suffices to resolve the nondeterminism (similar to abstraction of right context in LL parsing by fo-sets)

Example 9.6

$$
G: \quad \begin{array}{lcll} S' & \to & S & (1) \\ S & \to & B \mid C & (2,3) \\ B & \to & aB \mid b & (4,5) \\ C & \to & aC \mid c & (6,7) \end{array}
$$

NBA(G):

$$
\begin{array}{ll} & (ab, \varepsilon, \varepsilon) \\ \vdash & (b, a, \varepsilon) \\ \vdash & (\varepsilon, ab, \varepsilon) \\ \vdash & (\varepsilon, aB, 5) \\ \vdash & (\varepsilon, B, 54) \\ \vdash & (\varepsilon, S, 542) \\ \vdash & (\varepsilon, S', 5421) \end{array}
$$



$LR(0)$ Items and Sets II

Definition 9.7 ($LR(0)$ items and sets)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$ be start separated by $S' \to S$ and $S' \Rightarrow_r^* \alpha A w \Rightarrow_r \alpha\beta_1\beta_2 w$ (i.e., $A \to \beta_1\beta_2 \in P$).

- $[A \to \beta_1 \cdot \beta_2]$ is called an $LR(0)$ item for $\alpha\beta_1$.
- Given $\gamma \in X^*$, $LR(0)(\gamma)$ denotes the set of all $LR(0)$ items for γ, called the $LR(0)$ set (or: $LR(0)$ information) of γ.
- $LR(0)(G) := \{LR(0)(\gamma) \mid \gamma \in X^*\}$.

Corollary 9.8

1. *For every* $\gamma \in X^*$, $LR(0)(\gamma)$ *is finite.*
2. $LR(0)(G)$ *is finite.*
3. *The item* $[A \to \beta\cdot] \in LR(0)(\gamma)$ *indicates the possible reduction* $(w, \alpha\beta, z) \vdash (w, \alpha A, zi)$ *where* $\pi_i = A \to \beta$ *and* $\gamma = \alpha\beta$.
4. *The item* $[A \to \beta_1 \cdot Y\beta_2] \in LR(0)(\gamma)$ *indicates an incomplete handle* β_1 *(to be completed by shift operations or* ε*-reductions).*

$LR(0)$ Conflicts

Definition 9.9 ($LR(0)$ conflicts)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$ and $I \in LR(0)(G)$.

- I has a shift/reduce conflict if there exist $A \to \alpha_1 a \alpha_2, B \to \beta \in P$ such that
$$[A \to \alpha_1 \cdot a\alpha_2], [B \to \beta\cdot] \in I.$$
- I has a reduce/reduce conflict if there exist $A \to \alpha, B \to \beta \in P$ with $A \neq B$ or $\alpha \neq \beta$ such that
$$[A \to \alpha\cdot], [B \to \beta\cdot] \in I.$$

Lemma 9.10

$G \in LR(0)$ iff no $I \in LR(0)(G)$ contains conflicting items.

Proof.

omitted □

Computing $LR(0)$ Sets I

Theorem 9.11 (Computing $LR(0)$ sets)

Let $G = \langle N, \Sigma, P, S \rangle \in CFG_\Sigma$ *be start separated by* $S' \to S$ *and reduced.*

1. $LR(0)(\varepsilon)$ *is the least set such that*
   - $[S' \to \cdot S] \in LR(0)(\varepsilon)$ *and*
   - *if* $[A \to \cdot B\gamma] \in LR(0)(\varepsilon)$ *and* $B \to \beta \in P$, *then* $[B \to \cdot\beta] \in LR(0)(\varepsilon)$.
2. $LR(0)(\alpha Y)$ $(\alpha \in X^*, Y \in X)$ *is the least set such that*
   - *if* $[A \to \gamma_1 \cdot Y\gamma_2] \in LR(0)(\alpha)$, *then* $[A \to \gamma_1 Y \cdot \gamma_2] \in LR(0)(\alpha Y)$ *and*
   - *if* $[A \to \gamma_1 \cdot B\gamma_2] \in LR(0)(\alpha Y)$ *and* $B \to \beta \in P$, *then* $[B \to \cdot\beta] \in LR(0)(\alpha Y)$.

Computing $LR(0)$ Sets II

Example 9.12 (cf. Example 9.6)

$$
\begin{aligned}
G:\quad & S' \to S \\
& S \to B \mid C \\
& B \to aB \mid b \\
& C \to aC \mid c
\end{aligned}
\qquad [S' \to \cdot S] \in
$$

$$
LR(0)(\varepsilon)\quad \frac{[A \to \cdot B\gamma] \in LR(0)(\varepsilon),\, B \to \beta \in P}{\Longrightarrow\ [B \to \cdot\beta] \in LR(0)(\varepsilon)} \qquad \frac{[A \to \gamma_1 \cdot Y\gamma_2] \in LR(0)(\alpha)}{\Longrightarrow\ [A \to \gamma_1 Y \cdot \gamma_2] \in LR(0)(\alpha Y)}
$$

$I_0 := LR(0)(\varepsilon)$:	$[S' \to \cdot S]$	$[S \to \cdot B]$	$[S \to \cdot C]$	$[B \to \cdot aB]$
	$[B \to \cdot b]$	$[C \to \cdot aC]$	$[C \to \cdot c]$	
$I_1 := LR(0)(S)$:	$[S' \to S\cdot]$			
$I_2 := LR(0)(B)$:	$[S \to B\cdot]$			
$I_3 := LR(0)(C)$:	$[S \to C\cdot]$			
$I_4 := LR(0)(a)$:	$[B \to a \cdot B]$	$[C \to a \cdot C]$	$[B \to \cdot aB]$	$[B \to \cdot b]$
	$[C \to \cdot aC]$	$[C \to \cdot c]$		
$I_5 := LR(0)(b)$:	$[B \to b\cdot]$			
$I_6 := LR(0)(c)$:	$[C \to c\cdot]$			
$I_7 := LR(0)(aB)$:	$[B \to aB\cdot]$			
$I_8 := LR(0)(aC)$:	$[C \to aC\cdot]$			

($LR(0)(aa) = LR(0)(a) = I_4$, $LR(0)(ab) = LR(0)(b) = I_5$,
$LR(0)(ac) = LR(0)(c) = I_6$, $I_9 := LR(0)(\gamma) = \emptyset$ in all remaining cases)

Outline

Example 9.13

$$
\begin{array}{llll}
G: & S' & \to & S \\
 & S & \to & Aa \mid Bb \\
 & A & \to & a \\
 & B & \to & a
\end{array}
$$

$$
\begin{array}{lllll}
LR(0)(\varepsilon): & [S' \to \cdot S] & [S \to \cdot Aa] \quad [S \to \cdot Bb] & [A \to \cdot a] & [B \to \cdot a] \\
LR(0)(S): & [S' \to S \cdot] & & & \\
LR(0)(A): & [S \to A \cdot a] & & & \\
LR(0)(B): & [S \to B \cdot a] & & & \\
LR(0)(a): & [A \to a \cdot] & [B \to a \cdot] & & \\
LR(0)(Aa): & [S \to Aa \cdot] & & & \\
LR(0)(Ba): & [S \to Ba \cdot] & & &
\end{array}
$$

Note: G is unambiguous

Shift/Reduce Conflicts

Example 9.14

$$G: \quad S' \to S$$
$$\qquad S \to aS \mid a$$

$$LR(0)(\varepsilon): \quad [S' \to \cdot S] \quad [S \to \cdot aS] \quad [S \to \cdot a]$$
$$LR(0)(S): \quad [S' \to S \cdot]$$
$$LR(0)(a): \quad [S \to a \cdot S] \quad [S \to \cdot aS] \quad [S \to \cdot a] \quad [S \to a \cdot]$$
$$LR(0)(aS): \quad [S \to aS \cdot]$$

Note: G is unambiguous