Concurrency Theory

Strong bisimulation

Joost-Pieter Katoen and Thomas Noll

Lehrstuhl für Informatik 2
Software Modeling and Verification Group

http://www-i2.informatik.rwth-aachen.de/i2/ct13

December 12, 2013

RWTH AACHEN UNIVERSITY

Overview

Overview

Summary so far

1. Strong bisimulation is based on mutual mimicking each other
2. Strong bisimilarity $\sim$:
   - 2.1 is the largest strong bisimulation
   - 2.2 is an equivalence
   - 2.3 is a congruence (for CCS)
   - 2.4 is strictly finer than trace equivalence
   - 2.5 is deadlock sensitive

Aims of this lecture

1. Using games to show non-bisimilarity of two processes
2. Strong simulation: one-way bisimulation
3. Using fixed points to compute $\sim$

Strong bisimulation

Strong bisimulation [Park, 1981, Milner, 1989]

A binary relation $\mathcal{R} \subseteq Prc \times Prc$ is a strong bisimulation whenever for every $(P, Q) \in \mathcal{R}$, and $\alpha \in Act$:

1. if $P \xrightarrow{\alpha} P'$ then there exists $Q' \in Prc$ s.t. $Q \xrightarrow{\alpha} Q'$ and $(P', Q') \in \mathcal{R}$
2. if $Q \xrightarrow{\alpha} Q'$ then there exists $P' \in Prc$ s.t. $P \xrightarrow{\alpha} P'$ and $(P', Q') \in \mathcal{R}$.

Strong bisimilarity

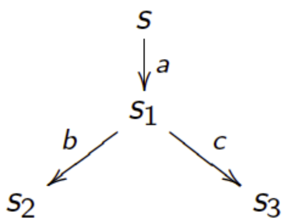
The processes P and Q are strongly bisimilar, denoted $P \sim Q$, iff there is a strong bisimulation $\mathcal{R}$ with $(P, Q) \in \mathcal{R}$. Thus,

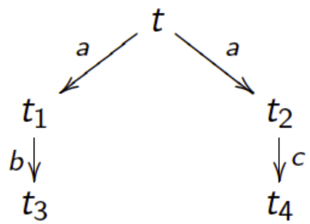
$$\sim \; = \bigcup \{ \mathcal{R} \mid \mathcal{R} \text{ is a strong bisimulation} \}.$$

Relation $\sim$ is called a strong bisimulation equivalence or strong bisimilarity.

Overview

How to show non-bisimilarity?





To prove that $s \not\sim t$

- Enumerate all binary relations and show that none of them containing (s, t) is a strong bisimulation. This is expensive, as there are 2^{k^2} binary relations on Prc with $|Prc| = k$.
- Make certain observations which will enable to disqualify many bisimulation candidates in one step.
- Use game characterization of strong bisimilarity.

Strong bisimulation game

Let $(Prc, Act, \longrightarrow)$ be an LTS and $s, t \in Prc$. Aim: does $s \sim t$?

We define a game with two players: an "attacker" and "defender".

- The game is played in rounds and configurations of the game are pairs of states from $Prc \times Prc$.
- In each round exactly one configuration is called current.
- Initially, the configuration (s, t) is the current one.

Intuition

The defender wants to show that $s \sim t$ while the attacker aims to show the opposite.

Rules of the bisimulation game

Game rules

In each round the current configuration (s, t) is changed as follows:

1. the attacker chooses one of the processes in the current configuration, say t, and makes an $\xrightarrow{\alpha}$-move for some $\alpha \in Act$ to t', say, and
2. the defender must respond by making an $\xrightarrow{\alpha}$-move in the other process s of the current configuration under the same action α, yielding $s \xrightarrow{\alpha} s'$.

The new pair of processes (s', t') becomes the current configuration.
The game continues with another round.

Game results

1. If one player cannot move, the other player wins.
2. If the game can be played *ad infinitum*, the defender wins.

Game characterization of bisimulation

Theorem [Stirling, 1995], [Thomas, 1993]

1. $s \sim t$ iff the defender has a universal winning strategy from configuration (s, t).
2. $s \not\sim t$ iff the attacker has a universal winning strategy from configuration (s, t).

(By means of as universal winning strategy, a player can always win, regardless of how the other player selects her moves.)

Proof.

Left as an exercise. □

A bisimulation game can be used to prove bisimilarity as well as non-bisimilarity.[1]
It often provides elegant arguments for $s \not\sim t$.

[1] In the next lectures, we will present yet another method to check this.

Example

A first example

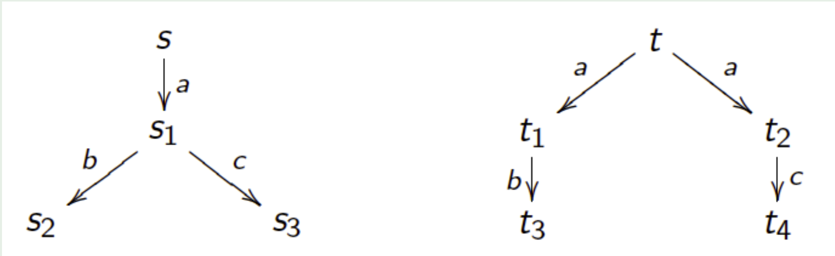
Use the game characterization to show $P \sim Q$ where:

$$\begin{aligned} P &= a.P_1 + a.P_2 \\ P_1 &= b.P_2 \\ P_2 &= b.P_2 \\ Q &= a.Q_1 \\ Q_1 &= b.Q_1. \end{aligned}$$

Example

Another example

Use the game characterization to show that $s \not\sim t$ where:



Overview

Strong simulation

Observation: sometimes, the concept of strong bisimulation is too strong (example: extending a system by new features).

Strong simulation

Relation $\mathcal{R} \subseteq Prc \times Prc$ is a strong simulation if, whenever $(P, Q) \in \mathcal{R}$ and $P \xrightarrow{\alpha} P'$, there exists $Q' \in Prc$ s.t. $Q \xrightarrow{\alpha} Q'$ and $(P', Q') \in \mathcal{R}$.

Q strongly simulates P, denoted $P \sqsubseteq Q$, if there exists a strong simulation $\mathcal{R}$ such that $(P, Q) \in \mathcal{R}$.

Thus: if Q strongly simulates P, then whatever transition P takes, Q can match it which retains all of P's options. But: P does not need to be able to match each transition of Q!

Simulation: example

Strong simulation

Relation $\mathcal{R} \subseteq Prc \times Prc$ is a strong simulation if, whenever $(P, Q) \in \mathcal{R}$ and $P \xrightarrow{\alpha} P'$, there exists $Q' \in Prc$ s.t. $Q \xrightarrow{\alpha} Q'$ and $(P', Q') \in \mathcal{R}$.

Q strongly simulates P, denoted $P \sqsubseteq Q$, if there exists a strong simulation $\mathcal{R}$ such that $(P, Q) \in \mathcal{R}$.

Example

$$\begin{array}{ccc} & P & \\ a \swarrow & & \searrow a \\ P_1 & & P_3 \\ b \downarrow & & \downarrow c \\ P_2 & & P_4 \end{array} \qquad \begin{array}{ccc} & Q & \\ & \downarrow a & \\ & Q_1 & \\ b \swarrow & & \searrow c \\ Q_2 & & Q_3 \end{array}$$

Q strongly simulates P, but not vice versa

This yields that:

$$a.b.\text{nil} + a.c.\text{nil} \sqsubseteq a.(b.\text{nil} + c.\text{nil})$$
$$a.(b.\text{nil} + c.\text{nil}) \not\sqsubseteq a.b.\text{nil} + a.c.\text{nil}.$$

Strong simulation and bisimilarity

Proposition

If $P \sim Q$, then $Q \sqsubseteq P$ and $P \sqsubseteq Q$.

Proof.

A strong bisimulation $\mathcal{R} \subseteq Prc \times Prc$ for $P \sim Q$ is a strong simulation for both directions. □

Caveat: the converse does generally not hold!

Example

$$\begin{array}{ccc} & P & \\ a \swarrow & & \searrow a \\ P_1 & & P_3 \\ \downarrow b & & \\ P_2 & & \end{array} \qquad \begin{array}{c} Q \\ \downarrow a \\ Q_1 \\ \downarrow b \\ Q_2 \end{array}$$

$P \sqsubseteq Q$ and $Q \sqsubseteq P$, but $P \not\sim Q$

Ready simulation

If $P \sqsubseteq Q$ and P has a deadlock, Q does not necessarily have a deadlock.

Ready simulation

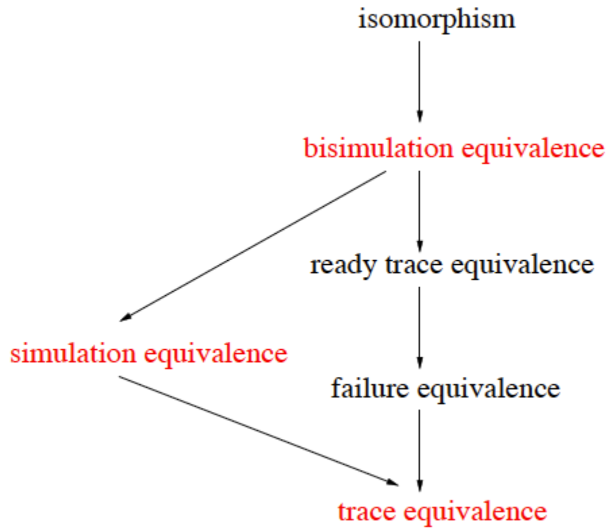
Relation $\mathcal{R} \subseteq Prc \times Prc$ is a ready simulation if, whenever $(P, Q) \in \mathcal{R}$ and $\alpha \in Act$:

1. if $P \xrightarrow{\alpha} P'$, then there exists $Q' \in Prc$ s.t. $Q \xrightarrow{\alpha} Q'$ and $(P', Q') \in \mathcal{R}$, and
2. if $Q \xrightarrow{\alpha}$, then $P \xrightarrow{\alpha}$.

Q ready simulates P, denoted $P \sqsubseteq_{rs} Q$, if there exists a ready simulation $\mathcal{R}$ such that $(P, Q) \in \mathcal{R}$.

In addition to the requirement for $\sqsubseteq$, ready simulation requires that if Q does not deadlock on α, then P neither does.

Overview of some behavioural equivalences



Overview

Strong bisimilarity

Recall: $\sim$ implies trace equivalence, and checking trace equivalence is PSPACE-complete.

What about checking $\sim$ between two processes?

Strong bisimilarity

$$\sim \; = \bigcup \{ \mathcal{R} \mid \mathcal{R} \text{ is a strong bisimulation} \}.$$

Note that $(2^{Prc \times Prc}, \subseteq)$ is a complete lattice[2] with $\bigcup$ and $\bigcap$ as least upper bound and greatest lower bound. We will show that $\sim$ can be characterized as a fixed point of a monotonic function on this lattice.

[2] Recall: a complete lattice is a partial order s.t. all its sets have a glb and a lub.

Fixed point characterization of $\sim$

Function on relations

Let $\mathcal{R} \subseteq Prc \times Prc$. Let $\mathcal{F} : 2^{Prc \times Prc} \to 2^{Prc \times Prc}$ be defined as follows: $(P, Q) \in \mathcal{F}(\mathcal{R})$ for all $P, Q \in Prc$ iff:

1. if $P \xrightarrow{\alpha} P'$ then there exists $Q' \in Prc$ s.t. $Q \xrightarrow{\alpha} Q'$ and $(P', Q') \in \mathcal{R}$
2. if $Q \xrightarrow{\alpha} Q'$ then there exists $P' \in Prc$ s.t. $P \xrightarrow{\alpha} P'$ and $(P', Q') \in \mathcal{R}$.

Intuition

$\mathcal{F}(\mathcal{R})$ contains all pairs of processes from which, in one round of the bisimulation game, the defender can ensure that the players reach a current configuration that is contained in $\mathcal{R}$.

Proposition

$\mathcal{R}$ is a strong bisimulation iff $\mathcal{R} \subseteq \mathcal{F}(\mathcal{R})$, and thus:

$$\sim \; = \bigcup \{ \mathcal{R} \in Prc \times Prc \mid \mathcal{R} \subseteq \mathcal{F}(\mathcal{R}) \}.$$

Computation of $\sim$

Proposition

$\mathcal{R}$ is a bisimulation iff $\mathcal{R} \subseteq \mathcal{F}(\mathcal{R})$, and thus:

$$\sim \; = \bigcup \{ \mathcal{R} \in Prc \times Prc \mid \mathcal{R} \subseteq \mathcal{F}(\mathcal{R}) \}.$$

Theorem

For finite-state process P with state space S, $\sim$ can be computed by:

$$\begin{aligned} \sim \; &= \; \bigcap_{i=0}^{\infty} \sim_i \quad \text{where } \sim_i \text{ is defined by} \\ \sim_0 \; &= \; S \times S \\ \sim_{i+1} \; &= \; \mathcal{F}(\sim_i). \end{aligned}$$

Proof.

Using the facts that $\mathcal{F}$ is monotonic on $(2^{Prc \times Prc}, \subseteq)$ and Taski's fixed point theorem. □

Naive algorithm for checking $\sim$

The fixed point characterisation suggests a polynomial-time algorithm.

Let P, Q be finite-state processes. Aim is to check whether $P \sim Q$.

1. Start with $\sim = (S_P \cup S_Q) \times (S_P \cup S_Q)$
2. If there is $(s, t) \in \sim$ with $s \xrightarrow{\alpha} s'$ and there is no t' with $t \xrightarrow{\alpha} t'$ and $(s', t') \in \sim$, then
   $\sim := \sim \setminus \{ (s, t) \}$
3. More efficient schemes do exist, but are not topic of this lecture

Complexity **[Balcázar et al., 1992]**

Deciding strong bisimilarity between finite LTSs is P-complete.

Overview

Summary

1. Checking (non-)bisimilarity can be done using a two-player game
2. Strong simulation is a one-way strong bisimulation
3. Strong simulation equivalence is strictly coarser than $\sim$
4. Ready simulation takes deadlocks into account
5. Strong bisimilarity can be characterised as a fixed point
6. This yields a polynomial-time procedure for determining $\sim$.