Modeling and Analysis of Hybrid Systems

Convex polyhedra

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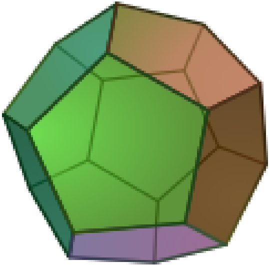
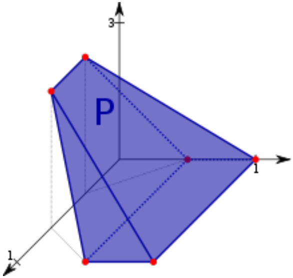
Contents

Polyhedra





Convex polyhedra

Definition

A (convex) polyhedron in $\mathbb{R}^d$ is the solution set to a finite number of linear inequalities with real coefficients in d real variables. A bounded polyhedron is called polytope.

Depending on the form of the representation, we distinguish between

- $\mathcal{H}$-polytopes and
- $\mathcal{V}$-polytopes.

$\mathcal{H}$-polytopes

Definition (Closed halfspace)

A d-dimensional closed halfspace is a set $\mathcal{H} = \{x \in \mathbb{R}^d \mid c \cdot x \leq z\}$ for some $c \in \mathbb{R}^d$, called the normal of the halfspace, and a $z \in \mathbb{R}$.

$\mathcal{H}$-polytopes

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Definition ($\mathcal{H}$-polyhedron, $\mathcal{H}$-polytope)

A d-dimensional $\mathcal{H}$-polyhedron $P = \bigcap_{i=1}^{n} \mathcal{H}_i$ is the intersection of finitely many closed halfspaces. A bounded $\mathcal{H}$-polyhedron is called an $\mathcal{H}$-polytope.

The facets of a d-dimensional $\mathcal{H}$-polytope are $d-1$-dimensional $\mathcal{H}$-polytopes.

$\mathcal{H}$-polytopes

An $\mathcal{H}$-polytope

$$P = \bigcap_{i=1}^{n} \mathcal{H}_i = \bigcap_{i=1}^{n} \{x \in \mathbb{R}^d \mid c_i \cdot x \leq z_i\}$$

can also be written in the form

$$P = \{x \in \mathbb{R}^d \mid Cx \leq z\}.$$

We call (C, z) the $\mathcal{H}$-representation of the polytope.

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Definition

A set S is called convex, if

$$\forall x, y \in S.\ \forall \lambda \in [0, 1] \subseteq \mathbb{R}.\ \lambda x + (1 - \lambda) y \in S.$$

$\mathcal{H}$-polyhedra are convex sets.

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$$CH(V) = \{x \in \mathbb{R}^d \mid \exists \lambda_1, \ldots, \lambda_n \in [0,1] \subseteq \mathbb{R}^d. \sum_{i=1}^{n} \lambda_i = 1 \wedge \sum_{i=1}^{n} \lambda_i v_i = x\}.$$

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Definition ($\mathcal{V}$-polytope)

A $\mathcal{V}$-polytope $P = CH(V)$ is the convex hull of a finite set $V \subset \mathbb{R}^d$. We call V the $\mathcal{V}$-representation of the polytope.

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Note that all $\mathcal{V}$-polytopes are bounded.

Motzkin's theorem

- For each $\mathcal{H}$-polytope, the convex hull of its vertices defines the same set in the form of a $\mathcal{V}$-polytope, and vice versa,
- each set defined as a $\mathcal{V}$-polytope can be also given as an $\mathcal{H}$-polytope by computing the halfspaces defined by its facets.

The translations between the $\mathcal{H}$- and the $\mathcal{V}$-representations of polytopes can be exponential in the state space dimension d.

Contents

Operations

If we represent reachable sets of hybrid automata by polytopes, we need some operations like

- membership computation,
- intersection, or the
- union of two polytopes.

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Alternatively: convert the $\mathcal{V}$-polytope into an $\mathcal{H}$-polytope by computing its facets.

Intersection

Intersection for two polytopes P_1 and P_2:

- $\mathcal{H}$-polytopes defined by $C_1 x \leq z_1$ and $C_2 x \leq z_2$:

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- $\mathcal{V}$-polytopes defined by V_1 and V_2:
  Convert P_1 and P_2 to $\mathcal{H}$-polytopes and convert the result back to a $\mathcal{V}$-polytope.

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- $\mathcal{V}$-polytopes defined by V_1 and V_2:
  $\mathcal{V}$-representation $V_1 \cup V_2$.
- $\mathcal{H}$-polytopes defined by $C_1 x \leq z_1$ and $C_2 x \leq z_2$:

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