

Modeling and Analysis of Hybrid Systems

Orthogonal polyhedra

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Informatik 2 - Theory of Hybrid Systems
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- We had a look at state set approximations by **convex polyhedra** and at basic operations (e.g., testing for membership or intersection) on these.
- Let us now have a look at another representation by **orthogonal polyhedra**.

Oliver Bournez, Oded Maler, and Amir Pnueli:
Orthogonal Polyhedra: Representation and Computation
Hybrid Systems: Computation and Control, LNCS 1569, pp. 46-60, 1999

1 Orthogonal polyhedra

■ Membership problem

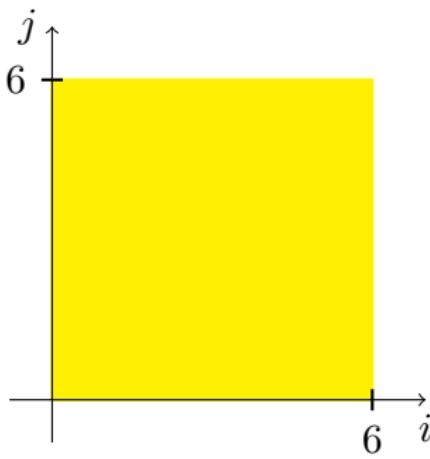
- Membership problem for the vertex representation
- Membership problem for the neighborhood representation
- Membership problem for the extreme vertex representation

■ Intersection

Definition

- **Domain**: bounded subset $X = [0, m]^d \subseteq \mathbb{R}^d$ ($m \in \mathbb{N}_+$) of the reals (can be extended to $X = \mathbb{R}_+^d$).
- **Elements of X** are denoted by $\mathbf{x} = (x_1, \dots, x_d)$, zero vector $\mathbf{0}$, unit vector $\mathbf{1}$.

$$X = [0, 6]^2$$

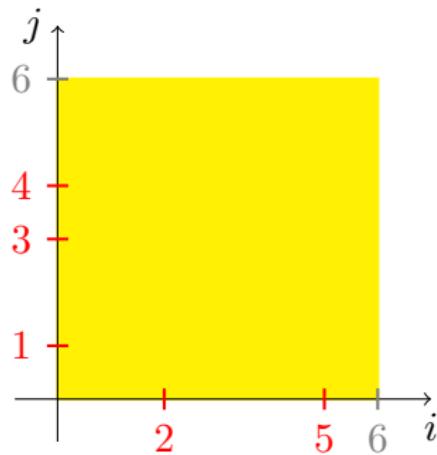


Definition

A d -dimensional grid associated with $X = [0, m]^d \subseteq \mathbb{R}^d$ ($m \in \mathbb{N}_+$) is a product of d subsets of $\{0, 1, \dots, m - 1\}$.

2-dimensional grid:

$$\{2, 5\} \times \{1, 3, 4\}$$



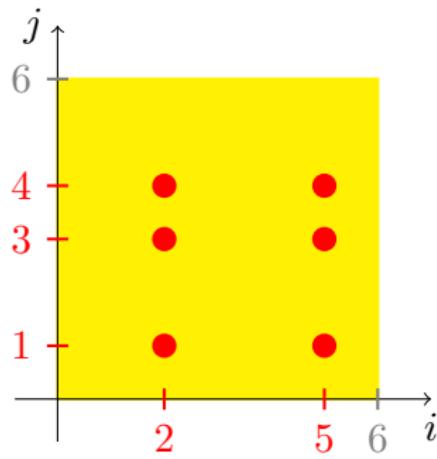
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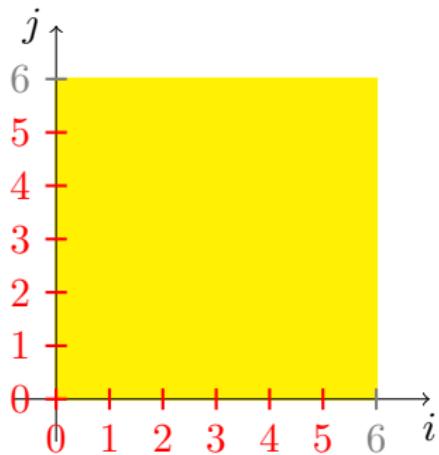


Grids

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The elementary grid associated with $X = [0, m]^d \subseteq \mathbb{R}^d$ ($m \in \mathbb{N}_+$) is $\mathbf{G} = \{0, 1, \dots, m - 1\}^d \subseteq \mathbb{N}^d$.

$$G = \{0, \dots, 5\} \times \{0, \dots, 5\}$$

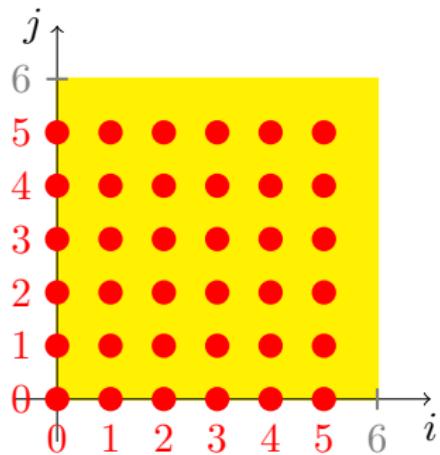


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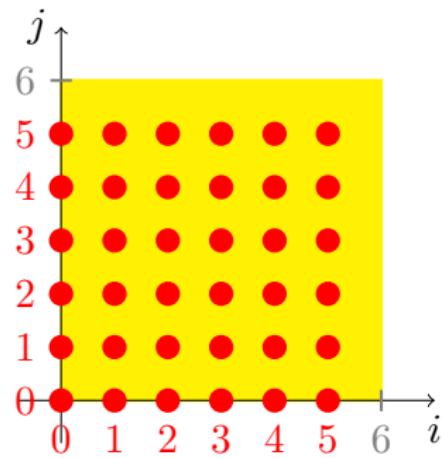
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Grids

The grid admits a natural **partial order** with $(m-1, \dots, m-1)$ on the top and **0** as bottom.

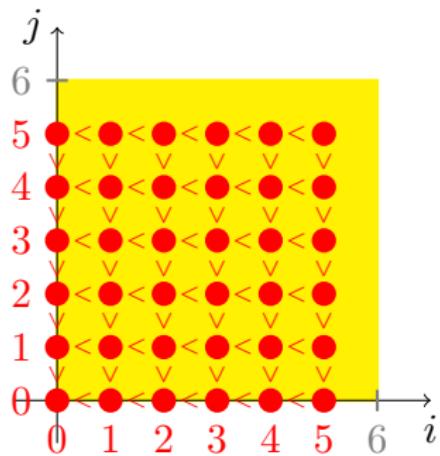
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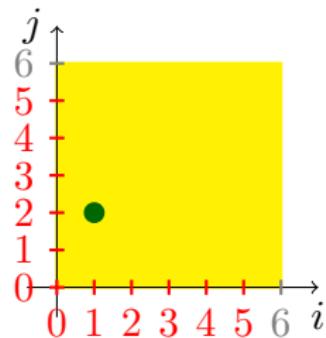
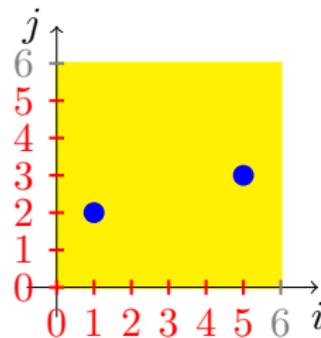
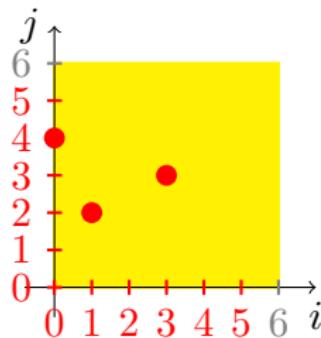


Grids

The set of subsets of the elementary grid \mathbf{G} forms a Boolean algebra $(2^{\mathbf{G}}, \cap, \cup, \sim)$ under the set-theoretic operations

- $A \cup B$
- $A \cap B$
- $\sim A = \mathbf{G} \setminus A$

for $A, B \subseteq \mathbf{G} \subset \mathbb{N}^d$.

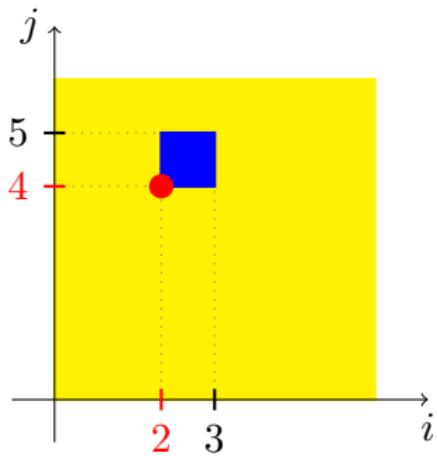


$$\{(0,4), (1,2), (3,3)\} \cap \{(1,2), (5,3)\} = \{(1,2)\}$$

Definition (Elementary box)

- The **elementary box** associated with a grid point $\mathbf{x} = (x_1, \dots, x_d)$ is $B(\mathbf{x}) = [x_1, x_1 + 1] \times \dots \times [x_d, x_d + 1]$.
- The point \mathbf{x} is called the **leftmost corner** of $B(\mathbf{x})$.
- The set of elementary boxes is denoted by **B**.

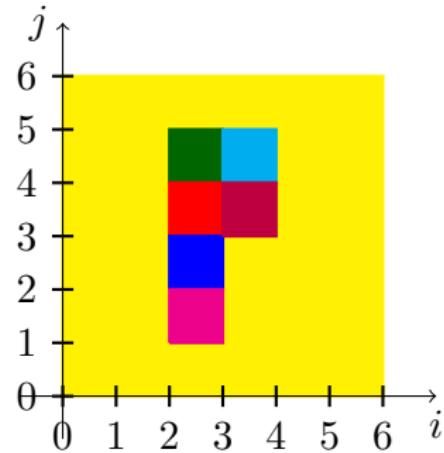
$$B((2, 4)) = [2, 3] \times [4, 5]$$



Definition (Orthogonal polyhedra)

An **orthogonal polyhedron** P is a union of elementary boxes, i.e., an element of 2^B .

$$\begin{aligned} & \{B((2, 4))\} \cup \{B((3, 4))\} \cup \\ & \{B((2, 3))\} \cup \{B((3, 3))\} \cup \\ & \{B((2, 2))\} \cup \\ & \{B((2, 1))\} \end{aligned}$$



Boolean algebra of orthogonal polyhedra

The set 2^B of orthogonal polyhedra is closed under the following operations:

- $A \sqcup B = A \cup B$
- $A \sqcap B = cl(int(A) \cap int(B))$
- $\neg A = cl(\sim A)$

with

- int the interior operator yielding the largest open set $int(A)$ contained in A , and
- cl the topological closure operator yielding the smallest closed set $cl(A)$ containing A .

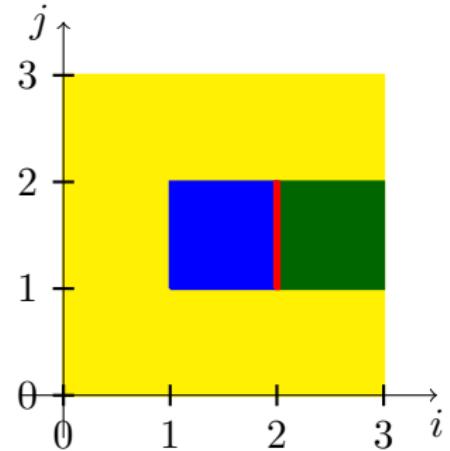
The set of orthogonal polyhedra forms a Boolean algebra $(2^B, \sqcap, \sqcup, \neg)$.

$$A \sqcap B = cl(int(A) \cap int(B))$$

$$([1, 2] \times [1, 2]) \sqcap ([2, 3] \times [1, 2]) =$$

$$cl(((1, 2) \times (1, 2)) \sqcap ((2, 3) \times (1, 2))) =$$

$$cl(\emptyset) = \emptyset$$



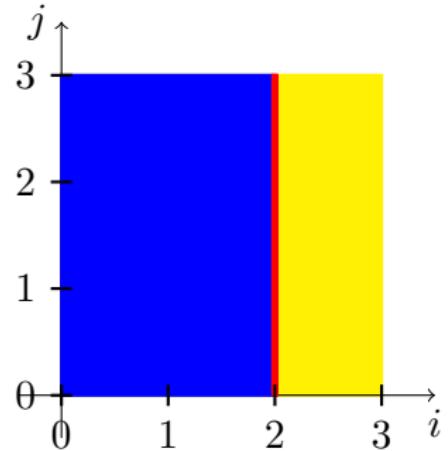
Note: $([1, 2] \times [1, 2]) \cap ([2, 3] \times [1, 2]) = [2, 2] \times [1, 2]$

$$\neg A = cl(\sim A)$$

$$\neg([0, 2] \times [0, 3]) =$$

$$cl(\sim ([0, 2] \times [0, 3])) =$$

$$cl((2, 3] \times [0, 3])) = [2, 3] \times [0, 3]$$

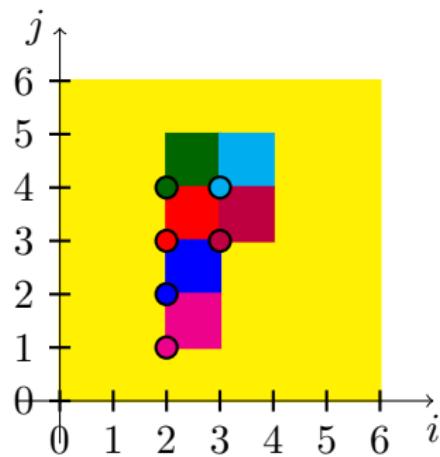


Note: $\sim ([0, 2] \times [0, 3]) = (2, 3] \times [0, 3]$

Connections

The bijection between \mathbf{G} and \mathbf{B} which associates every elementary box with its leftmost corner generates an isomorphism between $(2^{\mathbf{G}}, \cap, \cup, \sim)$ and $(2^{\mathbf{B}}, \sqcap, \sqcup, \neg)$.

Thus we can switch between point-based and box-based terminology according to what serves better the intuition.



Definition (Color function)

Let P be an orthogonal polyhedron. The color function $c : X \rightarrow \{0, 1\}$ is defined by

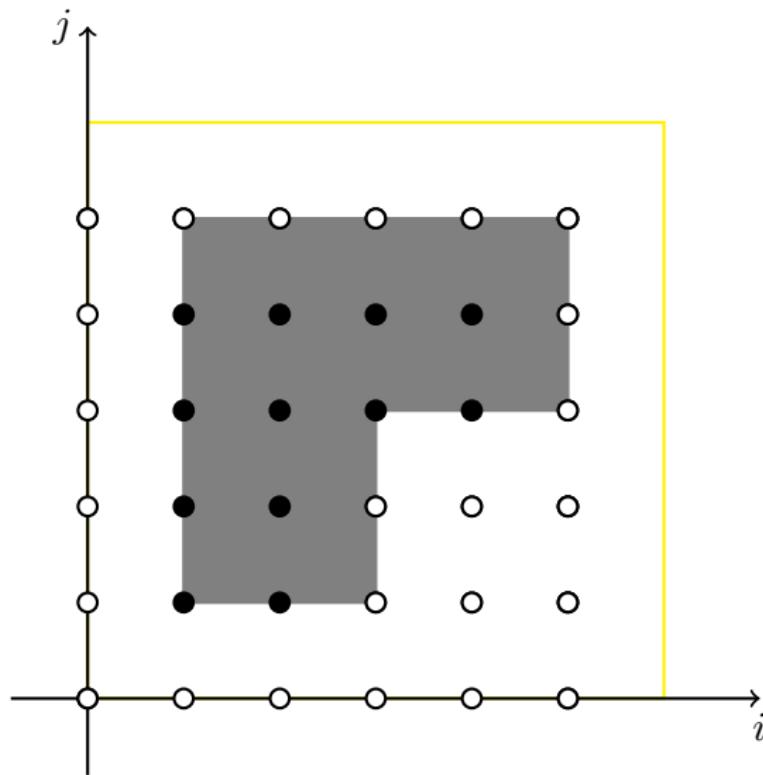
$$c(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \text{ is a grid point and } B(\mathbf{x}) \subseteq P \\ 0 & \text{otherwise} \end{cases}$$

for all $\mathbf{x} \in X$.

- If $c(\mathbf{x}) = 1$ we say that \mathbf{x} is **black** and that $B(\mathbf{x})$ is **full**.
- If $c(\mathbf{x}) = 0$ we say that \mathbf{x} is **white** and that $B(\mathbf{x})$ is **empty**.

Note that c almost coincides with the characteristic function of P as a subset of X . It differs from it only on right-boundary points.

Coloring



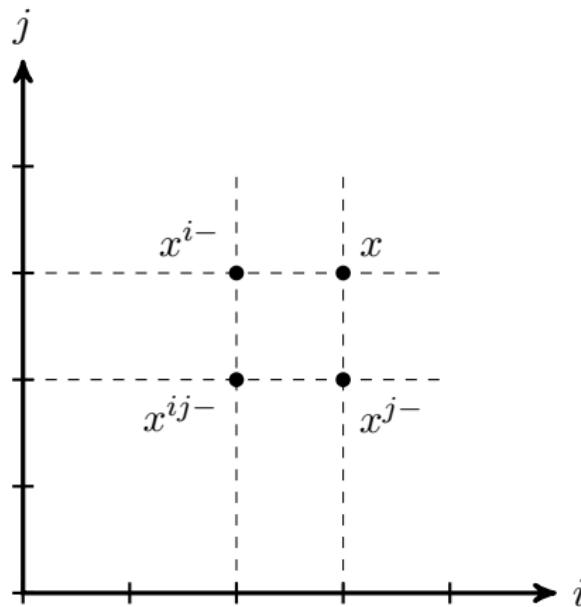
The following definitions capture the intuitive meaning of a facet and a vertex and, in particular, that the boundary of an orthogonal polyhedron is the union of its facets.

Definition (i -predecessor)

The i -predecessor of a grid point $\mathbf{x} = (x_1, \dots, x_d) \in X$ is

$\mathbf{x}^{i-} = (x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_d)$. We use \mathbf{x}^{ij-} to denote $(\mathbf{x}^{i-})^{j-}$.

When \mathbf{x} has no i -predecessor, we write \perp for the predecessor value.



Definition (Neighborhood)

The **neighborhood** of a grid point \mathbf{x} is the set

$$\mathcal{N}(\mathbf{x}) = \{x_1 - 1, x_1\} \times \dots \times \{x_d - 1, x_d\}$$

(the vertices of a box lying between $\mathbf{x} - 1$ and \mathbf{x}). For every i , $\mathcal{N}(\mathbf{x})$ can be partitioned into left and right i -neighborhoods

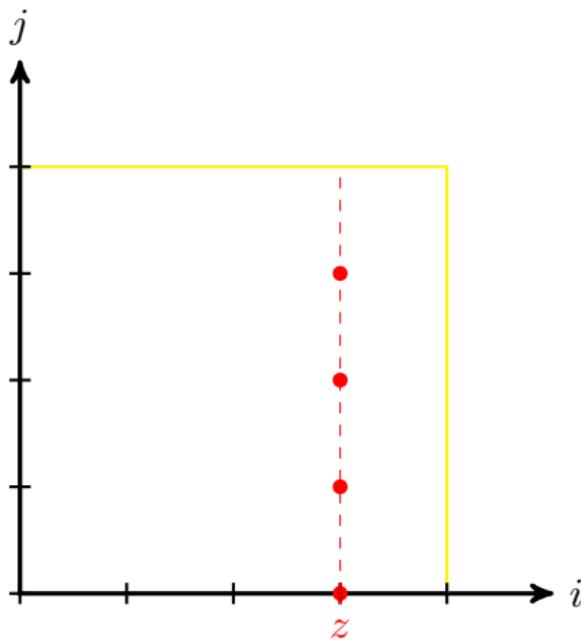
$$\mathcal{N}^{i-}(\mathbf{x}) = \{x_1 - 1, x_1\} \times \dots \times \{x_i - 1\} \times \{x_d - 1, x_d\}$$

and

$$\mathcal{N}^i(\mathbf{x}) = \{x_1 - 1, x_1\} \times \dots \times \{x_i\} \times \{x_d - 1, x_d\}.$$

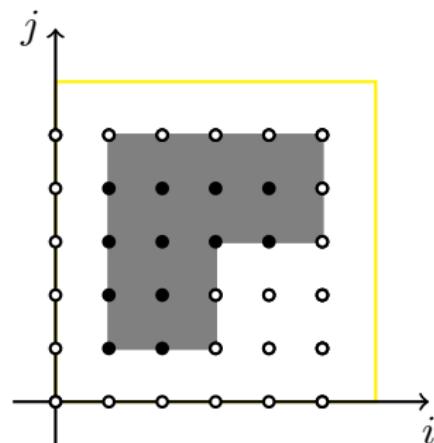
Definition (i -hyperplane)

An i -hyperplane is a $(d - 1)$ -dimensional subset $H_{i,z}$ of X consisting of all points \mathbf{x} satisfying $x_i = z$.



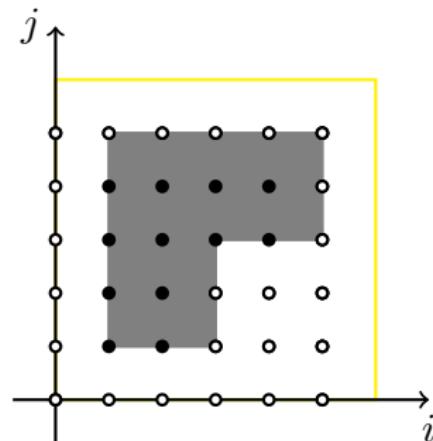
Observations:

- Facets are $d - 1$ -dimensional polyhedra.
- As such, facets are subsets of i -hyperplanes.
- The coloring changes on facets.
- White vertices need special care (closure to the “right”).



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Definition (i -facet)

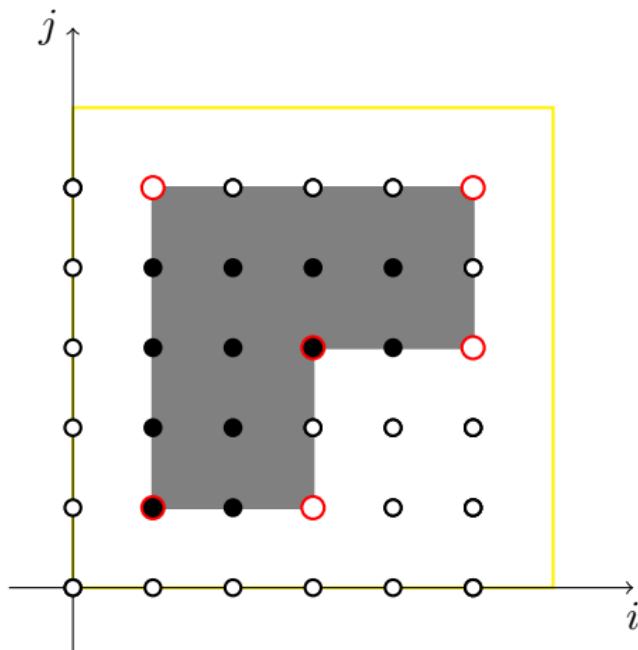
An i -facet of an orthogonal polyhedron P with color function c is

$$F_{i,z}(P) = \text{cl}\{\mathbf{x} \in H_{i,z} \mid c(\mathbf{x}) \neq c(\mathbf{x}^{i-})\}$$

for some integer $z \in [0, m)$.

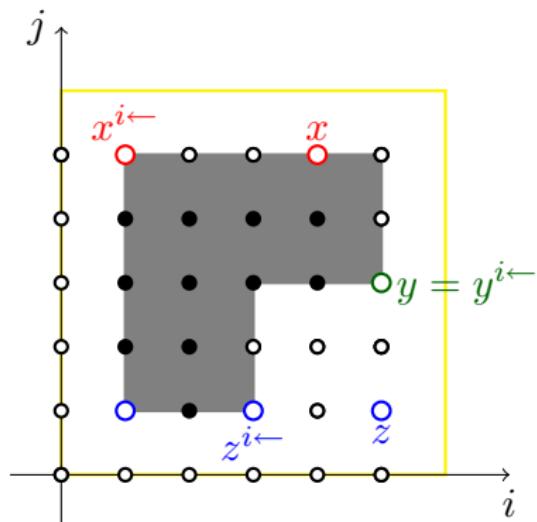
Definition (Vertex)

A **vertex** is a non-empty intersection of d distinct facets. The set of vertices of an orthogonal polyhedron P is denoted by $V(P)$.



Definition (i -vertex-predecessor)

- An i -vertex-predecessor of $\mathbf{x} = (x_1, \dots, x_d) \in X$ is a vertex of the form $(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_d)$ for some integer $z \in [0, x_i]$. When \mathbf{x} has no i -vertex-predecessor, we write \perp for its value.
- The first i -vertex-predecessor of \mathbf{x} , denoted by $x^{i\leftarrow}$, is the one with the maximal z .



A **representation scheme** for 2^B (2^G) is a set \mathcal{E} of syntactic objects such that there is a surjective function ϕ from \mathcal{E} to 2^B , i.e., every syntactic object represents at most one polyhedron and every polyhedron has at least one corresponding object.

If ϕ is an injection we say that the representation is **canonical**, i.e., every polyhedron has a unique representation.

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- **Vertex representation**: consists of the set $\{(\mathbf{x}, c(\mathbf{x})) \mid \mathbf{x} \text{ is a vertex}\}$, i.e., the vertices of P along with their color.
 - This representation is canonical.
 - The vertices alone is not a representation.
 - Not every set of points and colors is a valid representation of a polyhedron.

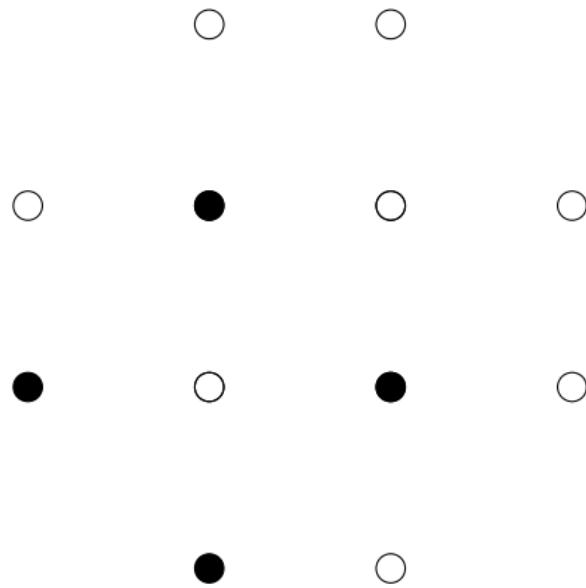
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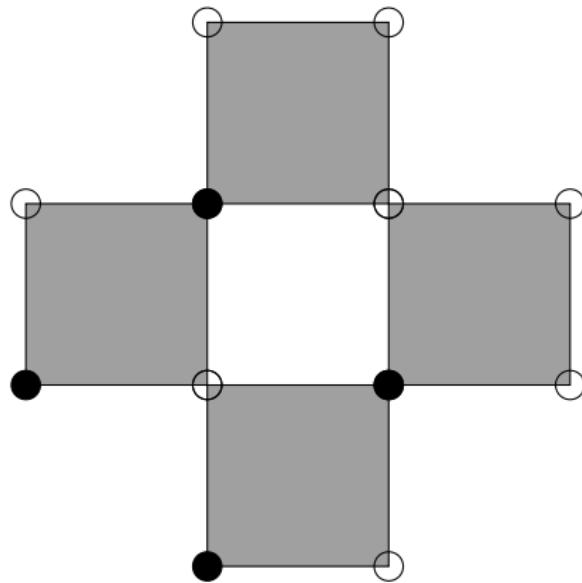
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- **Neighborhood representation**: the colors of all the 2^d points in the neighborhoods of the vertices is attached as additional information.
- **Extreme vertex representation**: instead of maintaining all the neighborhood of each vertex, it suffices to keep only the *parity* of the number of black points in that neighborhood. In fact, it suffices to keep only vertices with odd parity.

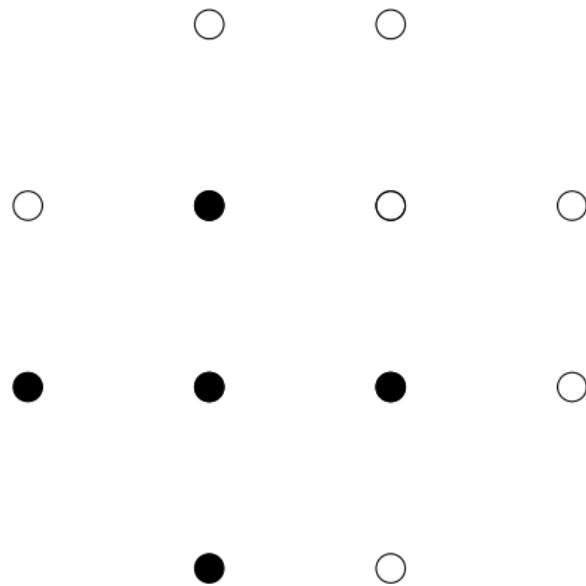
Vertex representation



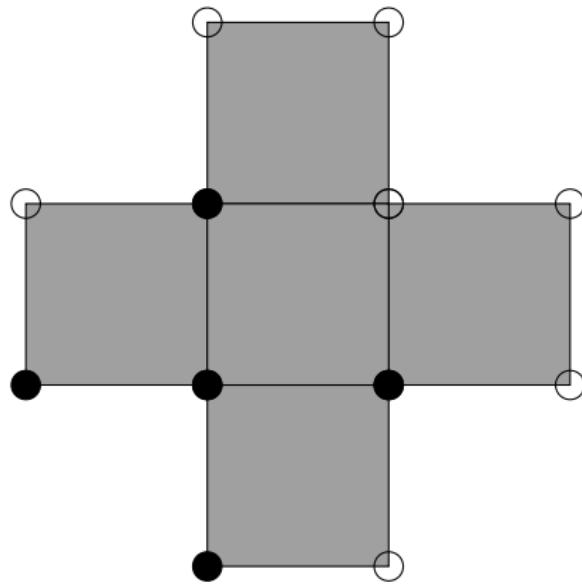
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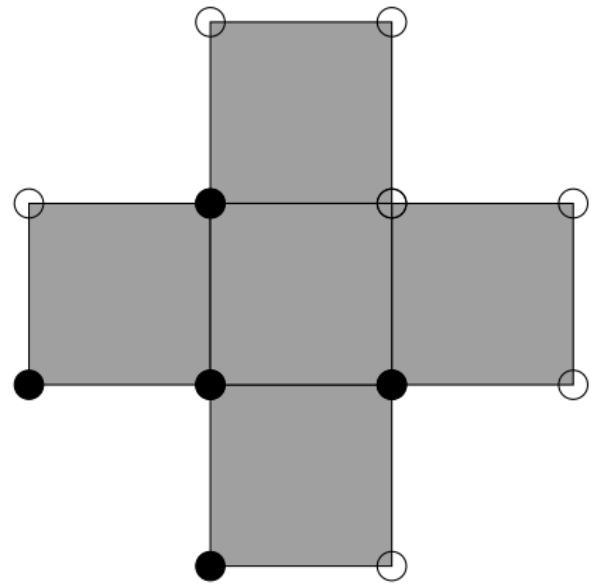
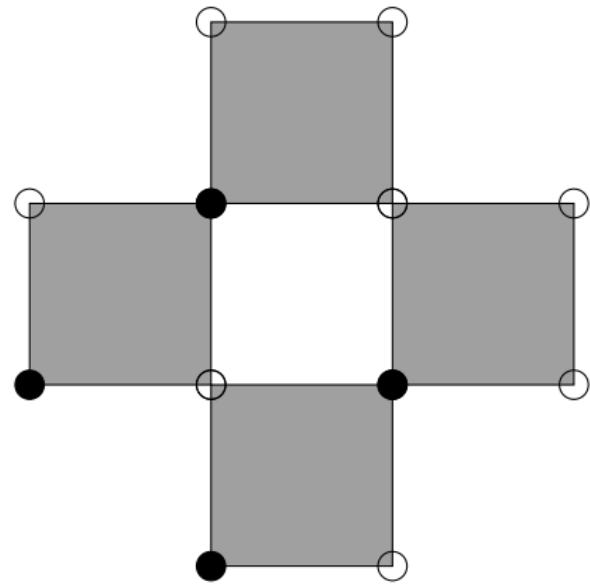
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1 Orthogonal polyhedra

■ Membership problem

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- Membership problem for the extreme vertex representation

■ Intersection

Membership problem

The membership problem

Given a representation of a polyhedron P and a grid point \mathbf{x} , determine $c(\mathbf{x})$, that is, whether $B(\mathbf{x}) \subseteq P$.

1 Orthogonal polyhedra

■ Membership problem

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■ Intersection

Observations

- A point \mathbf{x} is on an i -facet iff

$$\exists \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}). c(\mathbf{x}'^{i-}) \neq c(\mathbf{x}').$$

- A point \mathbf{x} is a vertex iff

$$\forall i \in \{1, \dots, d\}. \exists \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}). c(\mathbf{x}'^{i-}) \neq c(\mathbf{x}').$$

- A point \mathbf{x} is not a vertex iff

$$\exists i \in \{1, \dots, d\}. \forall \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}). c(\mathbf{x}'^{i-}) = c(\mathbf{x}').$$

Example

For $d = 2$ and $\mathbf{x} = (x_1, x_2)$ it means:

- \mathbf{x} is on a 1-facet iff

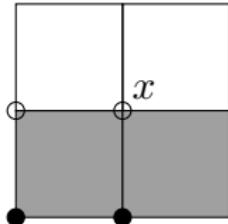
$$c(x_1 - 1, x_2 - 1) \neq c(x_1, x_2 - 1) \vee c(x_1 - 1, x_2) \neq c(x_1, x_2).$$

- \mathbf{x} is on a 2-facet iff

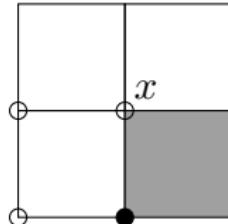
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- \mathbf{x} is a vertex iff both of the above hold.
- \mathbf{x} is not a vertex iff one of the above does not hold.

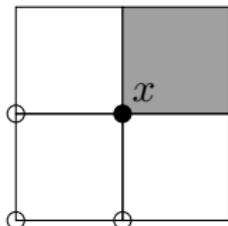
Example



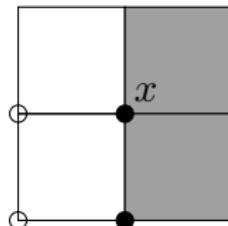
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$$c(x_1 - 1, x_2 - 1) \neq c(x_1, x_2 - 1)$$



$$c(x_1, x_2 - 1) \neq c(x_1, x_2) \wedge \\ c(x_1 - 1, x_2) \neq c(x_1, x_2)$$



$$c(x_1, x_2 - 1) \neq c(x_1, x_2) \wedge$$

Lemma (Color of a non-vertex)

Let \mathbf{x} be a non-vertex. Then there exists a direction $j \in \{1, \dots, d\}$ such that

$$\forall \mathbf{x}' \in \mathcal{N}^j(\mathbf{x}) \setminus \{\mathbf{x}\}. c(\mathbf{x}'^{j-}) = c(\mathbf{x}').$$

Let j be such a direction. Then $c(\mathbf{x}) = c(\mathbf{x}^{j-})$.

Color computation

Lemma (Color of a non-vertex)

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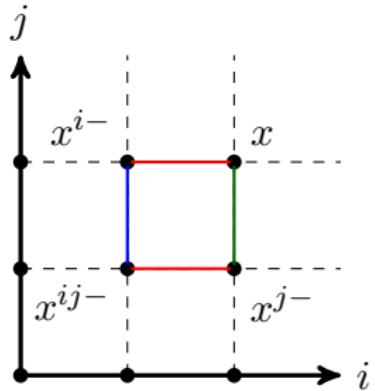
Proof. A point \mathbf{x} is not a vertex iff

$$\exists i \in \{1, \dots, d\}. \forall \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}). c(\mathbf{x}'^{i-}) = c(\mathbf{x}').$$

Thus j always exists. Let i and j be two dimensions satisfying the above requirements.

Case 1: $j = i$: Straightforward

Case 2: $j \neq i$: For i we have $c(\mathbf{x}^{i-}) = c(\mathbf{x})$ and $c(\mathbf{x}^{ij-}) = c(\mathbf{x}^{j-})$. For j we have $c(\mathbf{x}^{ij-}) = c(\mathbf{x}^{i-})$. Thus $c(\mathbf{x}) = c(\mathbf{x}^{j-})$.



Complexity

Consequently we can calculate the color of a non-vertex \mathbf{x} based on the color of all points in $\mathcal{N}(\mathbf{x}) - \{\mathbf{x}\}$: just find some j satisfying the conditions of the above lemma and let $c(\mathbf{x}) = c(\mathbf{x}^{j-})$.

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Theorem

The membership problem for vertex representation can be solved in time $\mathcal{O}(n^d d 2^d)$ using space $\mathcal{O}(n^d)$.

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- We must recursively determine the color of at most n^d grid points.
- For each of them we must check at most d dimensions if they satisfy the condition of the lemma on the color of a non-vertex.
- Checking the condition invokes $2^d - 1$ color comparisons.

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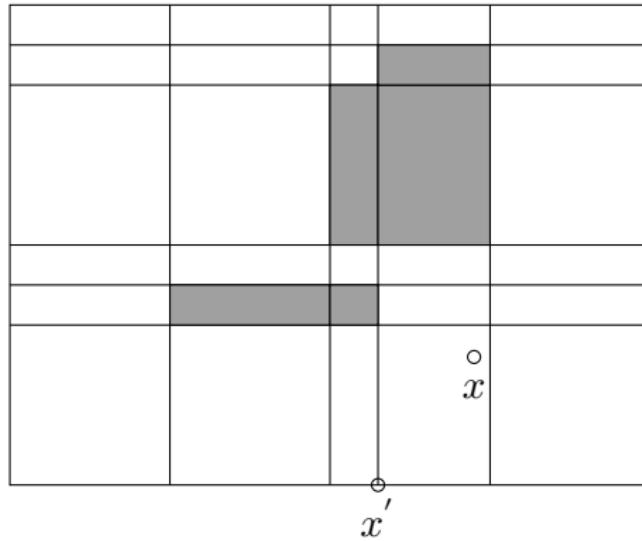
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- We must recursively determine the color of at most n^d grid points.
- For each of them we must check at most d dimensions if they satisfy the condition of the lemma on the color of a non-vertex.
- Checking the condition invokes $2^d - 1$ color comparisons.

However, this algorithm is not very efficient, because in the worst-case one has to calculate the color of all the grid points between $\mathbf{0}$ and \mathbf{x} .

Induced grid

We can improve it using the notion of an **induced grid**: let the i -scale of P be the set of the i -coordinates of the vertices of P , and let the induced grid be the Cartesian product of its i -scales.



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- Every rectangle in the induced grid has a uniform color.
- Calculating the color of a point reduces to finding its closest “dominating” point on the induced grid and applying the algorithm to that grid in $\mathcal{O}(n^d d 2^d)$ time.

1 Orthogonal polyhedra

■ Membership problem

- Membership problem for the vertex representation
- Membership problem for the neighborhood representation
- Membership problem for the extreme vertex representation

■ Intersection

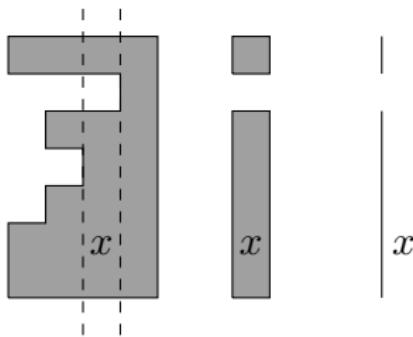
Membership problem for the neighborhood representation

We introduce an $\mathcal{O}(n \log n)$ membership algorithm for the neighborhood representation, based on successive projections of P into polyhedra of smaller dimension.

Definition (i -slice and i -section)

Let P be an orthogonal polyhedron and z an integer in $[0, m)$.

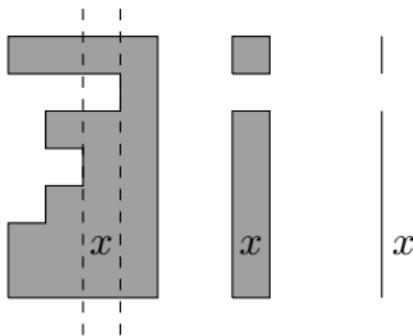
- The *i -slice of P at z* is the d -dimensional orthogonal polyhedron $J_{i,z}(P) = P \cap \{\mathbf{x} \mid z \leq x_i \leq z + 1\}$.
- The *i -section of P at z* is the $(d - 1)$ -dimensional orthogonal polyhedron $\mathcal{J}_{i,z}(P) = J_{i,z}(P) \cap H_{i,z}$.



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Clearly, the membership of $\mathbf{x} = (x_1, \dots, x_d)$ can be reduced into membership in $\mathcal{J}_{i,x_i}(P)$, which is a $(d - 1)$ -dimensional problem. By successively reducing dimensionality for every i we obtain a point whose color is that of \mathbf{x} .

Calculating the i -sections for the neighborhood representation

How can the main computational activity, the calculation of i -sections, be done using the neighborhood representation?

Lemma (Vertex of a section)

Let P be an orthogonal polyhedron and let P' be its i -section at $x_i = z$. A point \mathbf{x} is a vertex of P' iff $\mathbf{y} = \mathbf{x}^{i\leftarrow} \neq \perp$ and for every $j \neq i$ there exists $\mathbf{x}' \in \mathcal{N}^i(\mathbf{y}) \cap \mathcal{N}^j(\mathbf{y})$ such that $c(\mathbf{x}'^{j-}) \neq c(\mathbf{x}')$.

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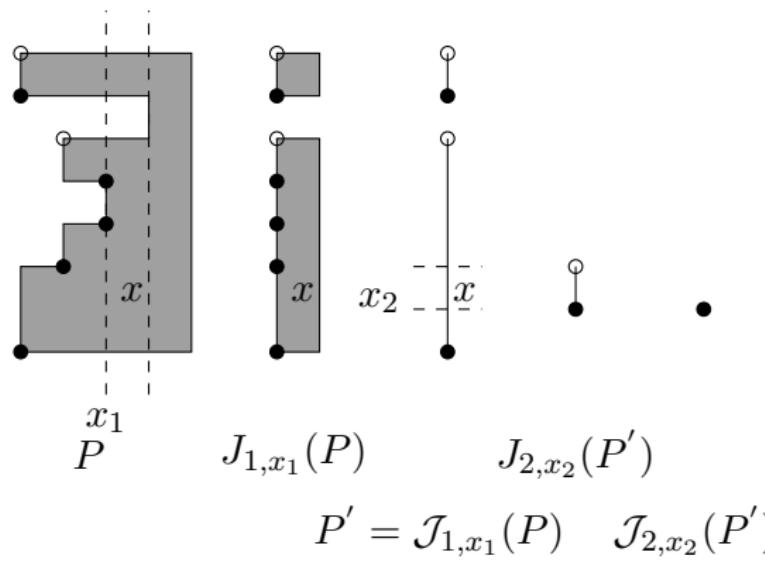
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- Hence it takes $\mathcal{O}(nd(\log n + 2^d))$ to get rid of one dimension.
- This is repeated d times until p is contracted into a point.

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- We use $\text{parity}(\mathbf{x})$ to denote the parity of the number of black points in $\mathcal{N}(\mathbf{x})$.
- A point \mathbf{x} is said to be **extreme** if $\text{parity}(\mathbf{x}) = 1$.

Lemma

An extreme point is a vertex.

Proof: By induction on the dimension d . The base case $d = 1$ is immediate. For $d > 1$, choose $i \in \{1, \dots, d\}$. Exactly one of $\mathcal{N}^{i-}(\mathbf{x})$ and $\mathcal{N}^i(\mathbf{x})$ contains an odd number of black points. Assume w.l.o.g. that it is $\mathcal{N}^i(\mathbf{x})$. By induction hypothesis \mathbf{x} is a vertex in $\mathcal{J}_{i,x_i}(P)$. I.e., for every $j \neq i$ there exists $\mathbf{x}' \in \mathcal{N}^j(\mathbf{x})$ such that $c(\mathbf{x}'^{j-}) \neq c(\mathbf{x}')$. Since one cannot have $c(\mathbf{x}') = c(\mathbf{x}'^{i-})$ for all $\mathbf{x}' \in \mathcal{N}^i(\mathbf{x})$, \mathbf{x} is a vertex of P .

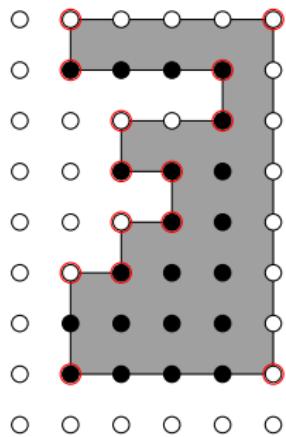
The converse is not true, i.e., vertices need not be extreme.

- An **extreme vertex representation** consists in representing an orthogonal polyhedron by the set of its extreme vertices. (Additionally, the color of the origin is stored in a bit. From this information the colors of all extreme vertices can be inferred.)

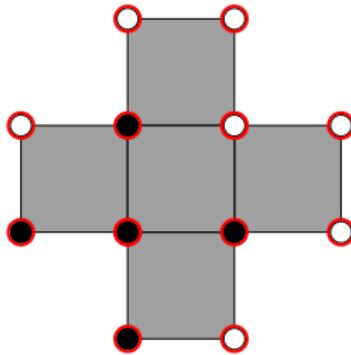
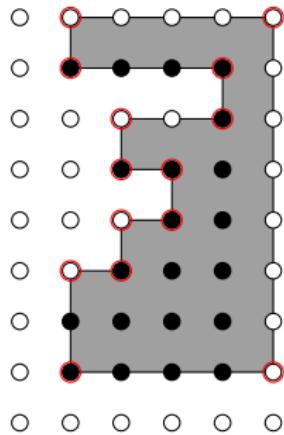
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- Examples (extreme vertices are marked red):

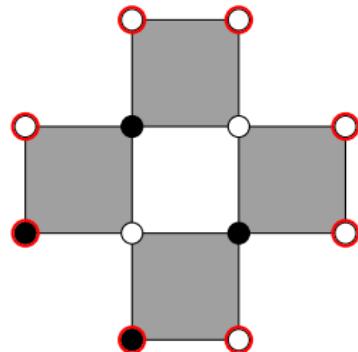
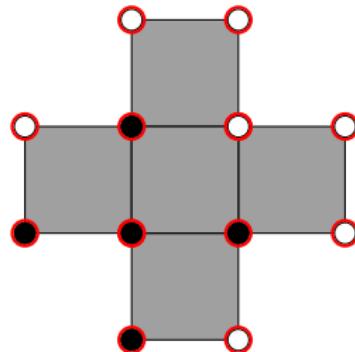
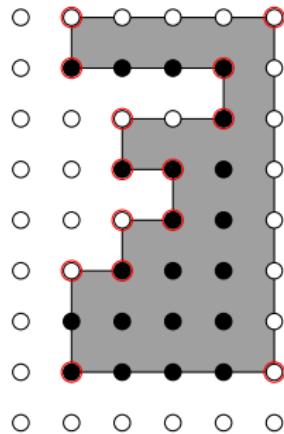
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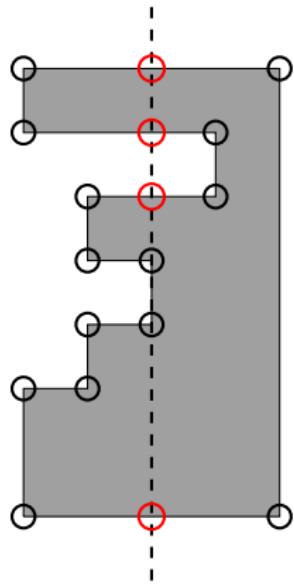
The membership problem is solved again by projection. For that we need again a rule to determine which points of an i -section are extreme vertices.

Lemma (Extreme vertices of a section)

Let P be an orthogonal polyhedron and let $P' = \mathcal{J}_{i,z}(P)$. A point \mathbf{x} is an extreme vertex of P' iff it has an odd number of extreme i -vertex-predecessors.

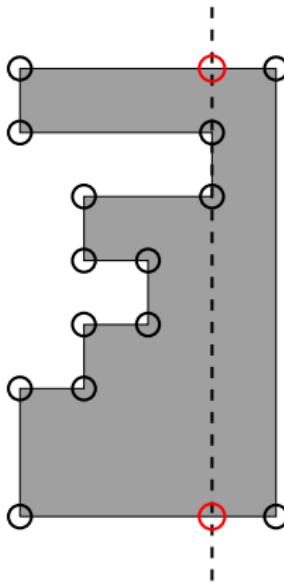
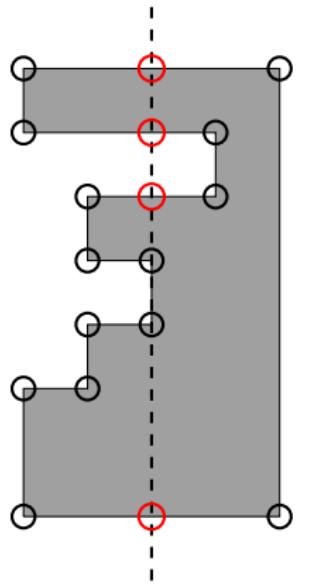
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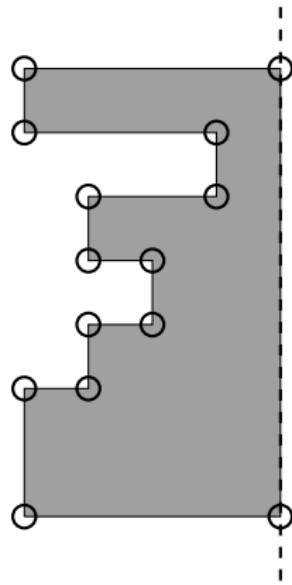
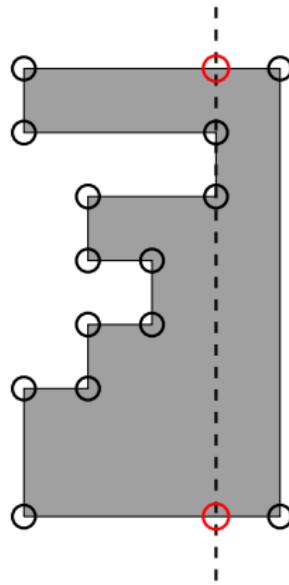
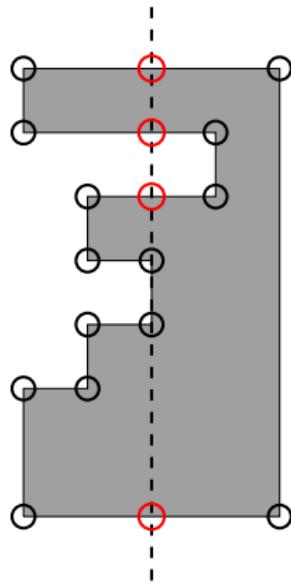
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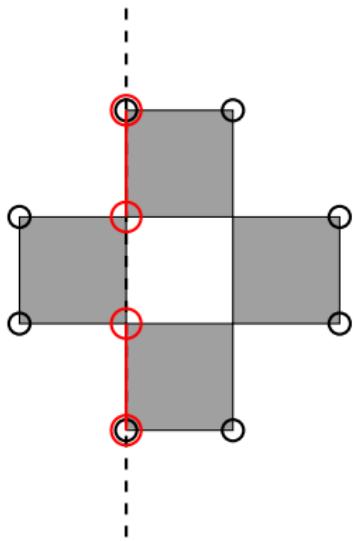
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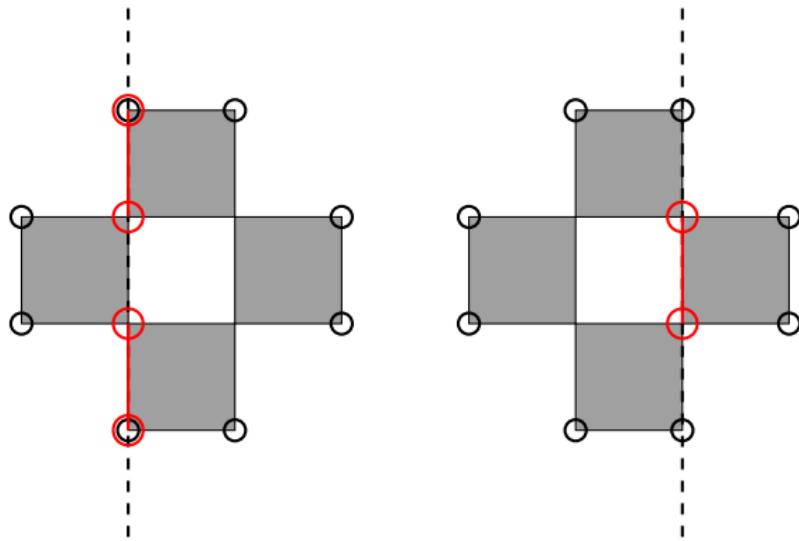
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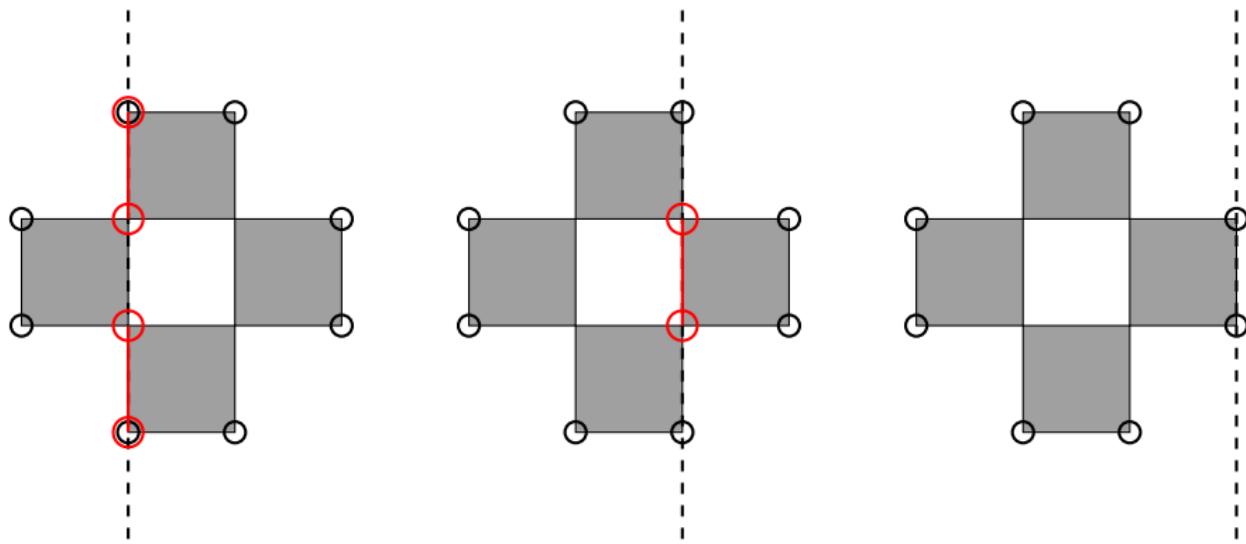
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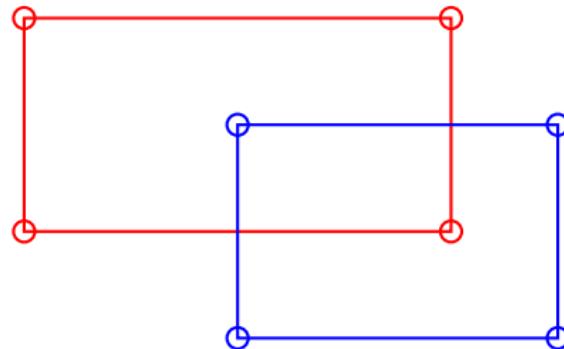
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Intersection

We assume two polyhedra P_1 and P_2 with n_1 and n_2 vertices, respectively. After intersection some vertices disappear and some new vertices are created.



Lemma

A point x is a vertex of $P_1 \cap P_2$ only if for every dimension i , x is on an i -facet of P_1 or on an i -facet of P_2 .

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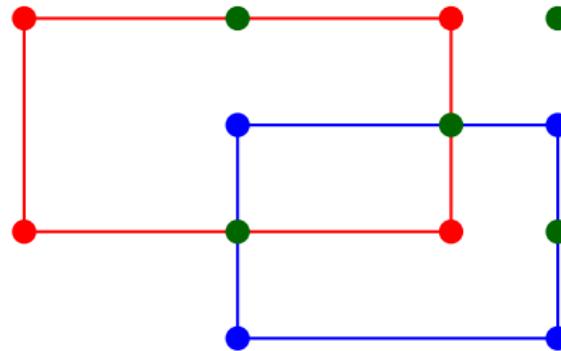
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whose number is not greater than $n_1 + n_2 + n_1 n_2$.

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 - Use the vertex rules to determine, whether the point is a vertex of the intersection.

Intersection example: Vertex representation

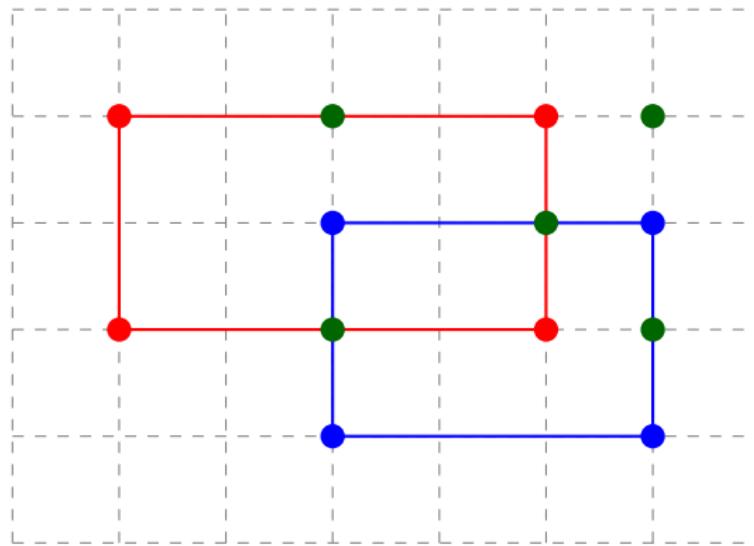
Vertex rule: A point \mathbf{x} is a vertex iff

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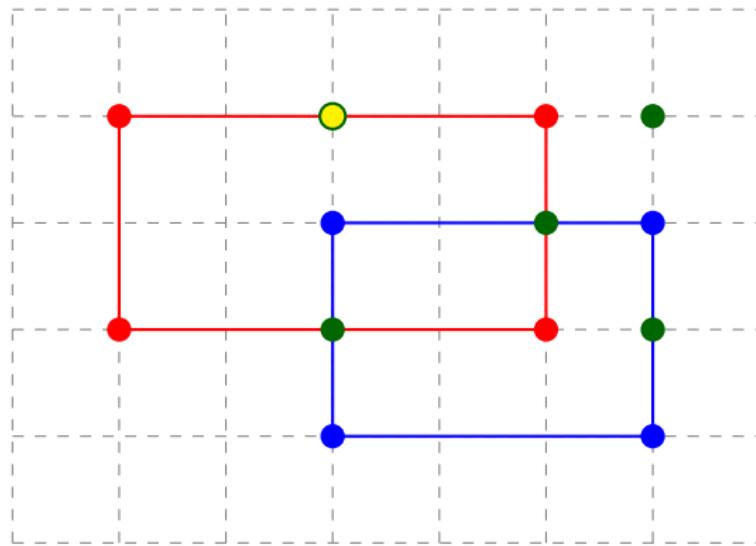
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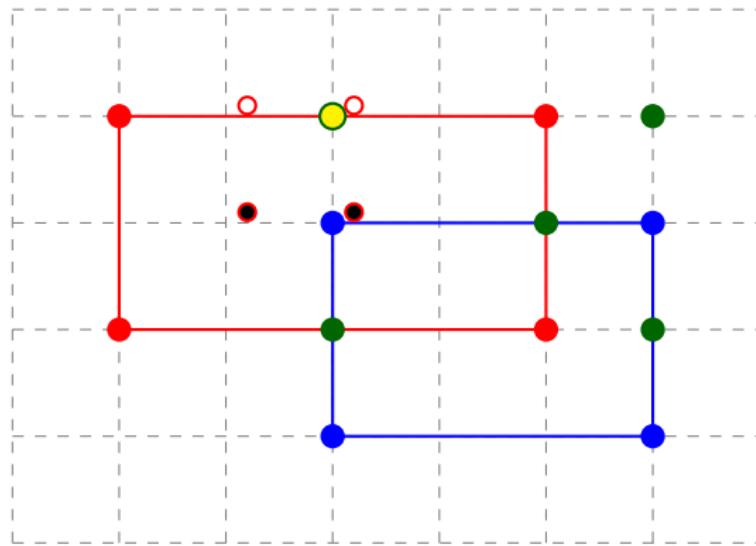
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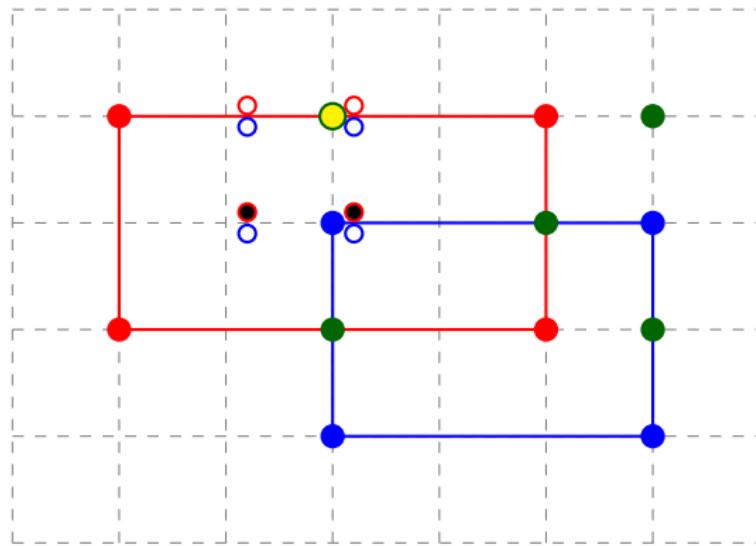
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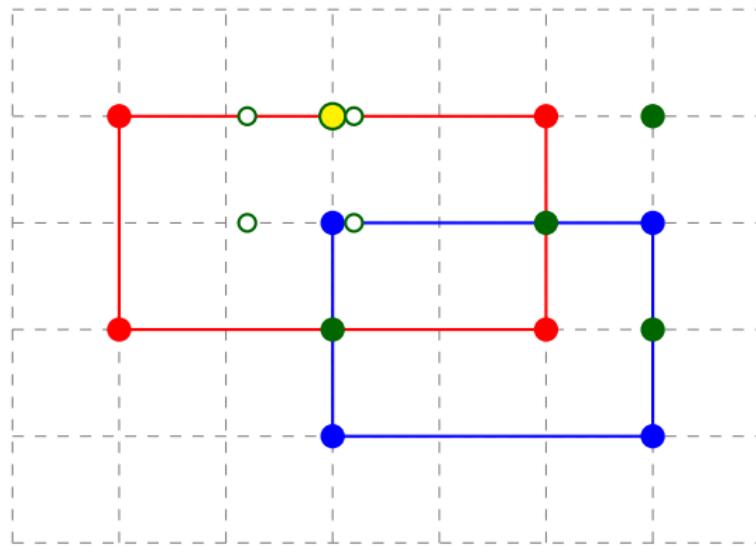
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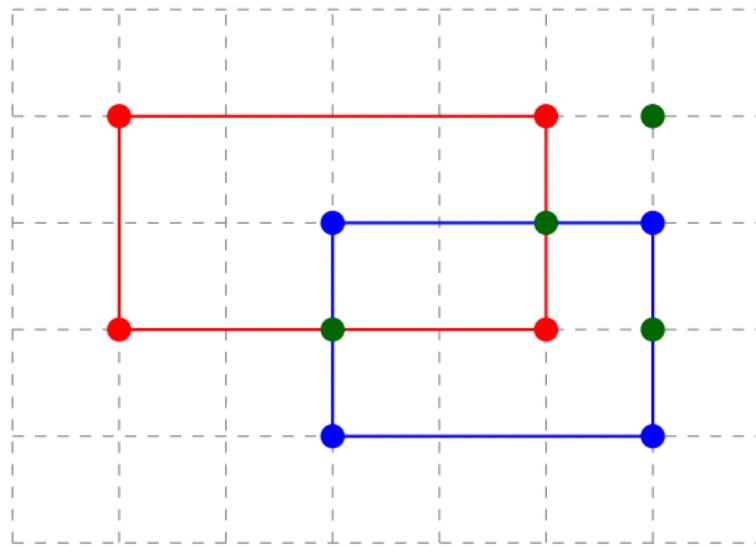
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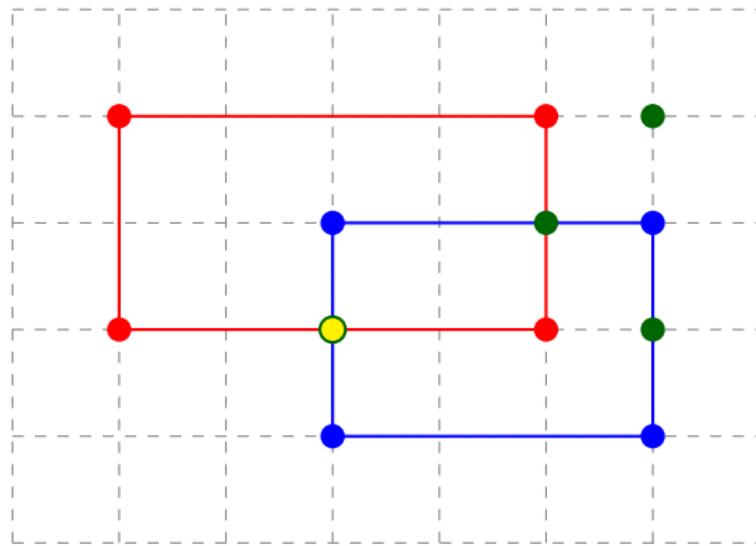
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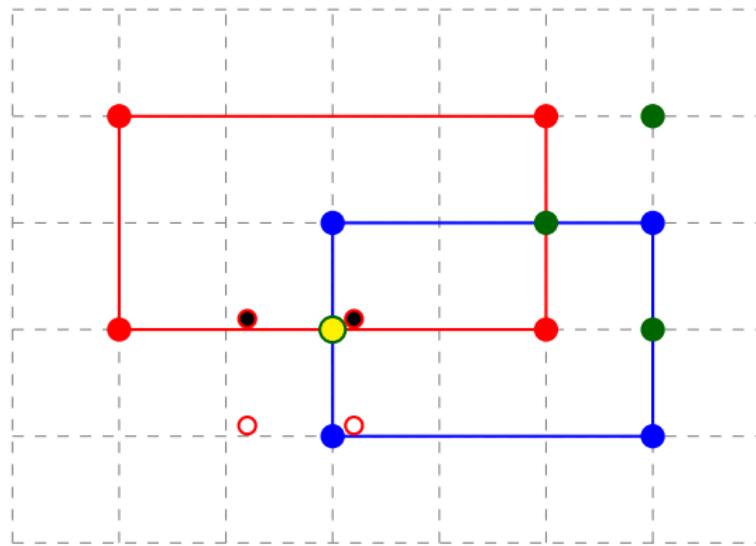
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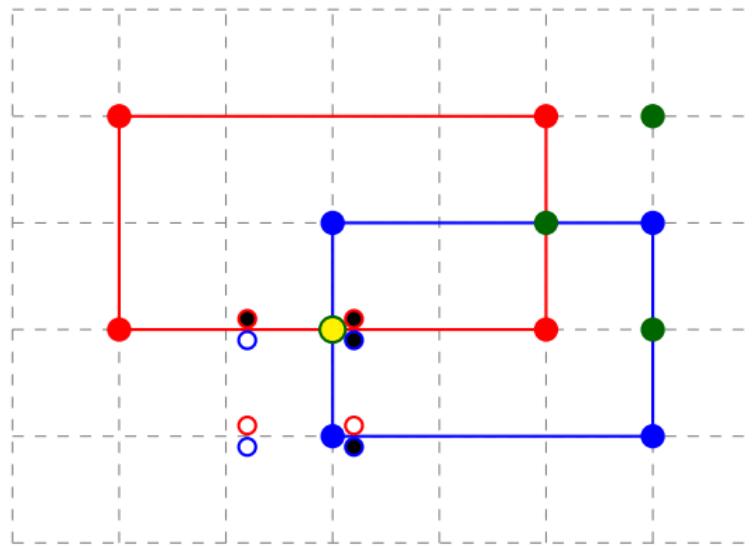
$$\forall i \in \{1, \dots, d\}. \exists \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}). c(\mathbf{x}'^{i-}) \neq c(\mathbf{x}').$$



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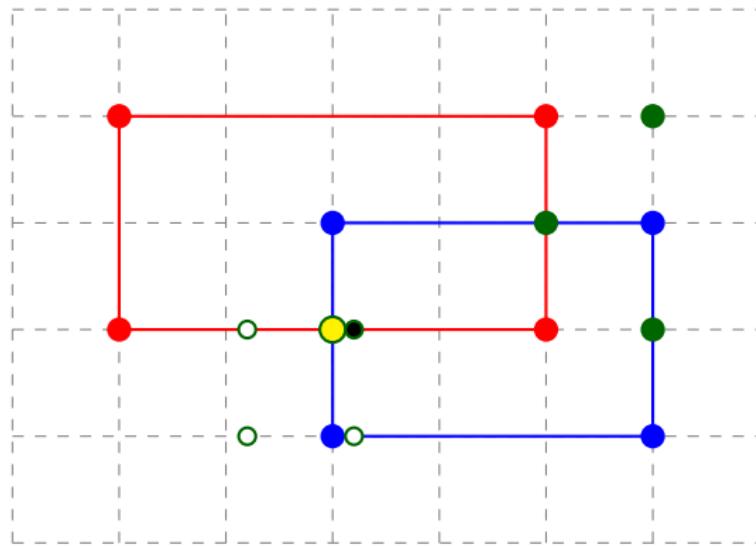
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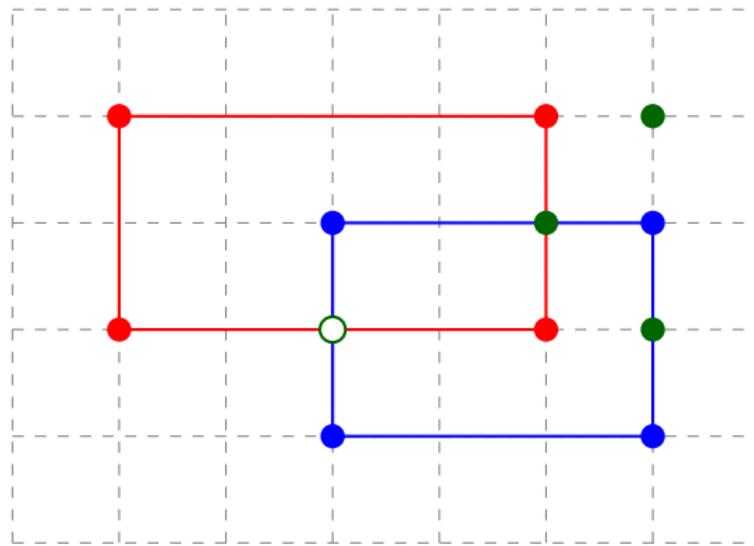
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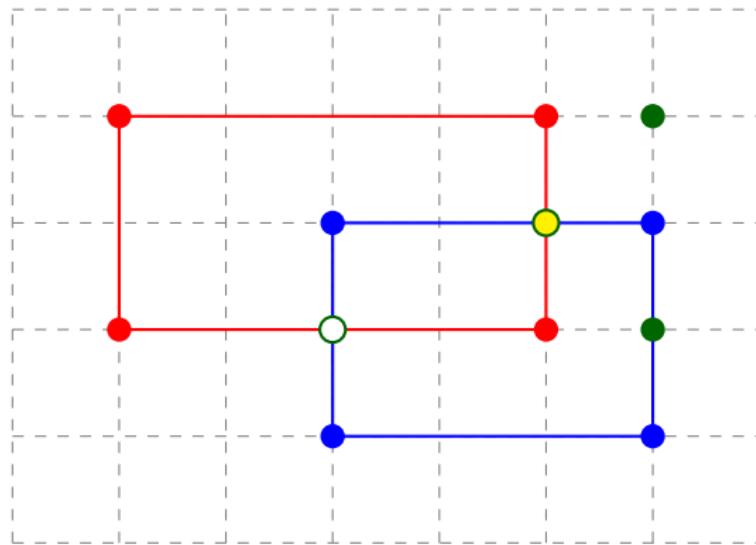
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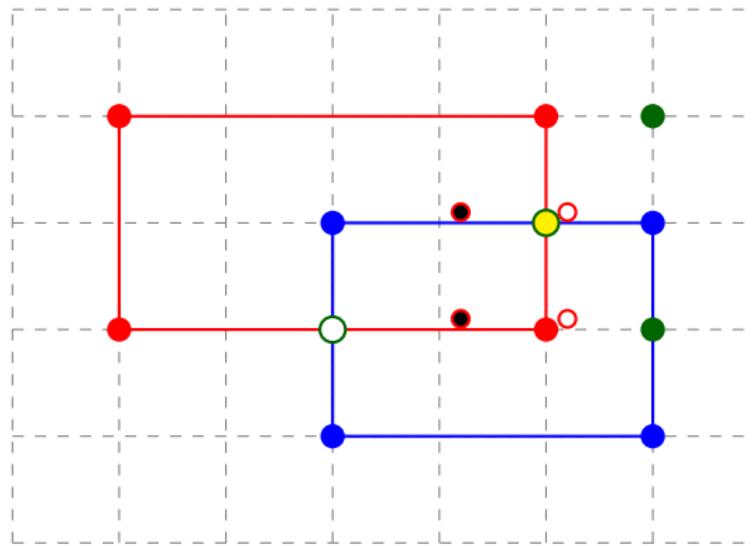
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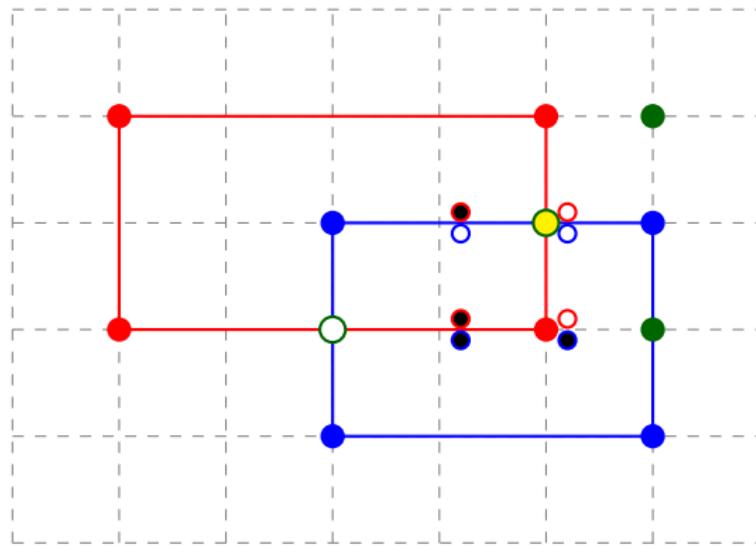
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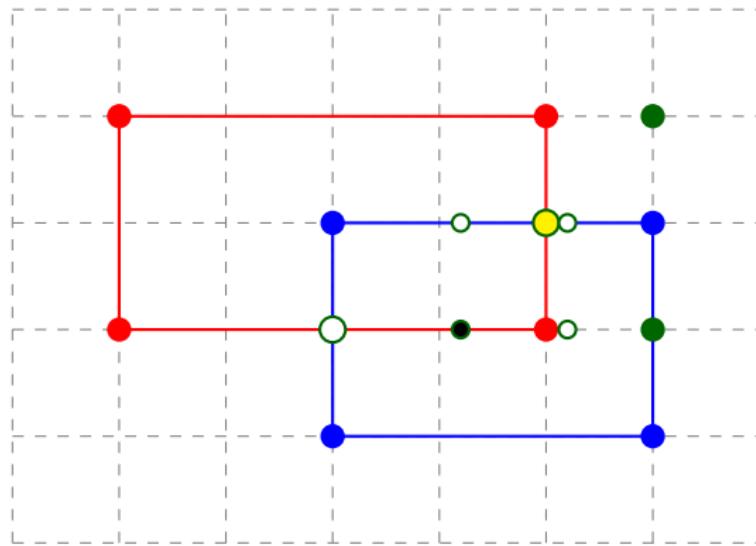
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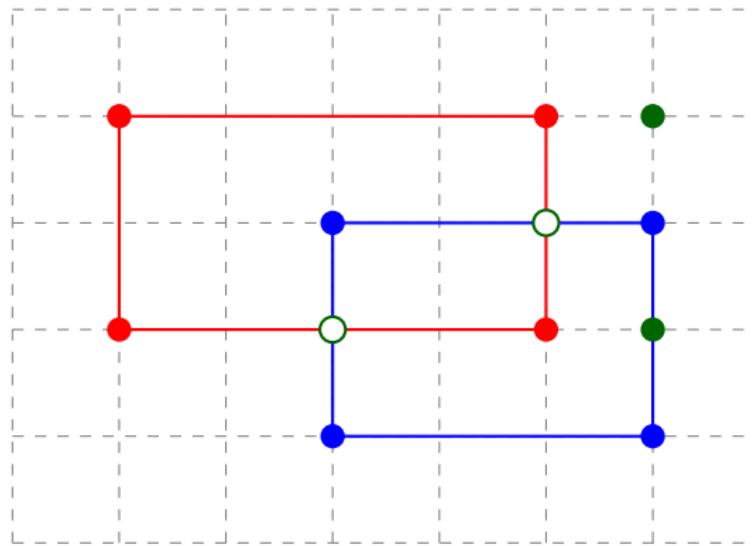
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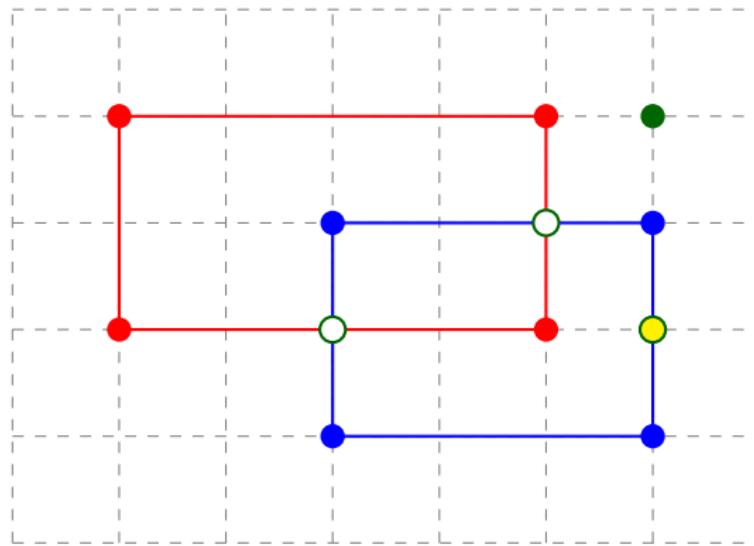
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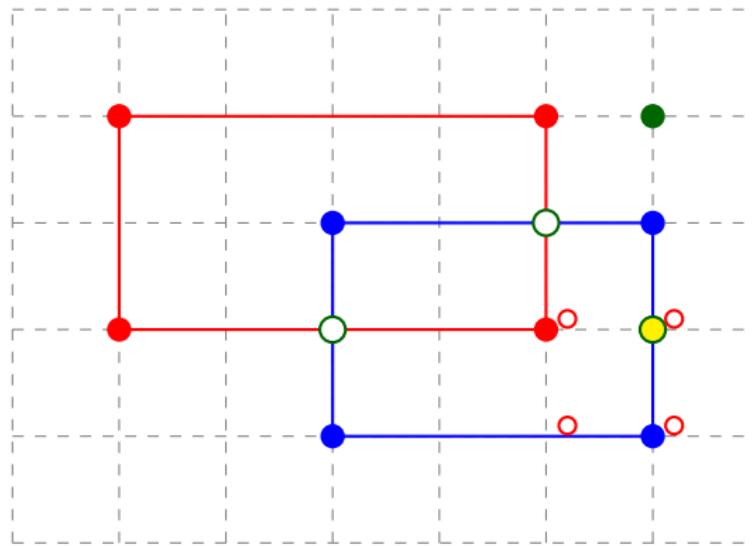
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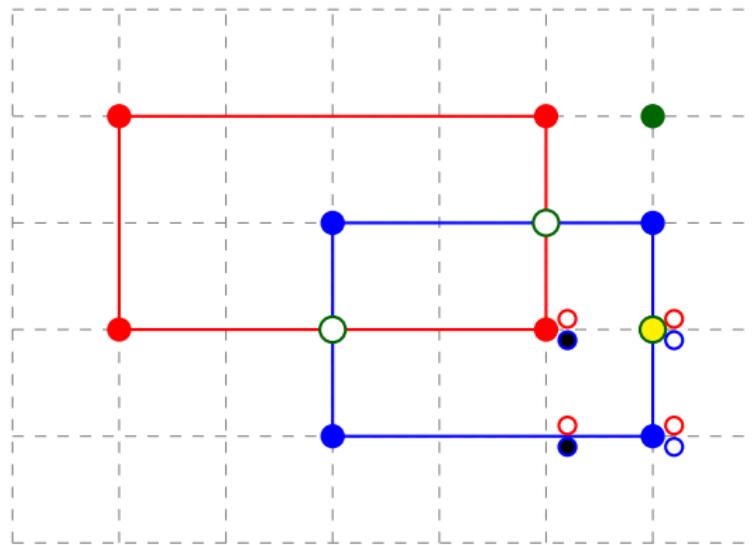
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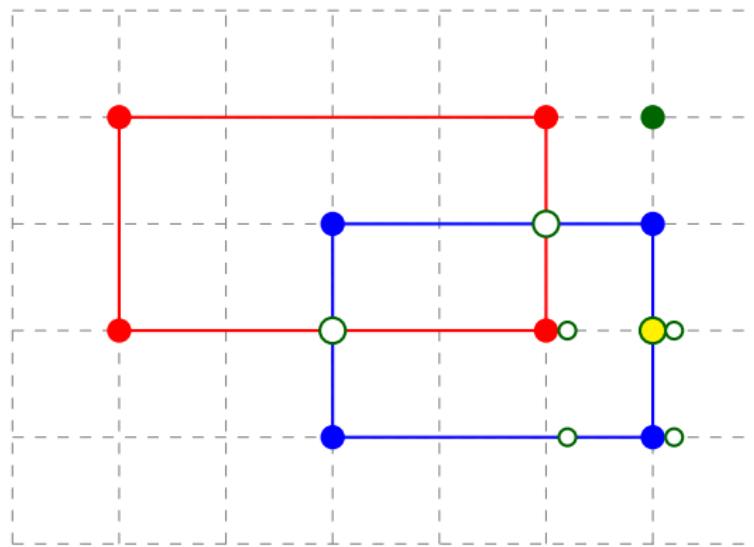
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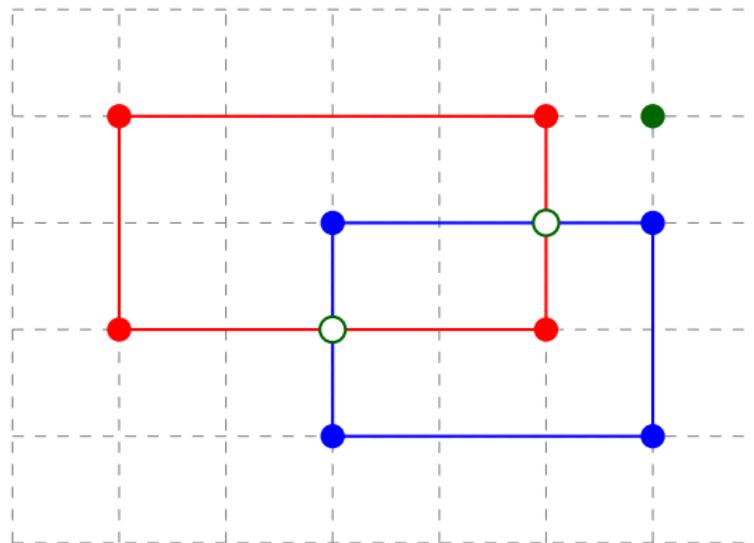
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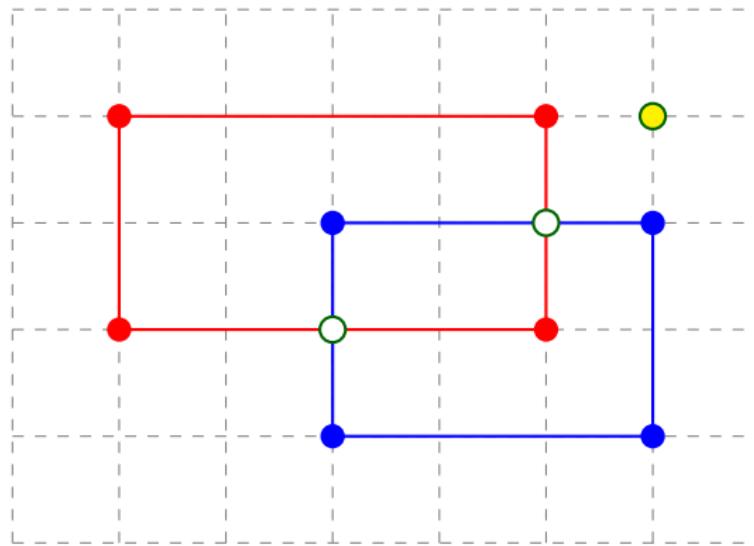
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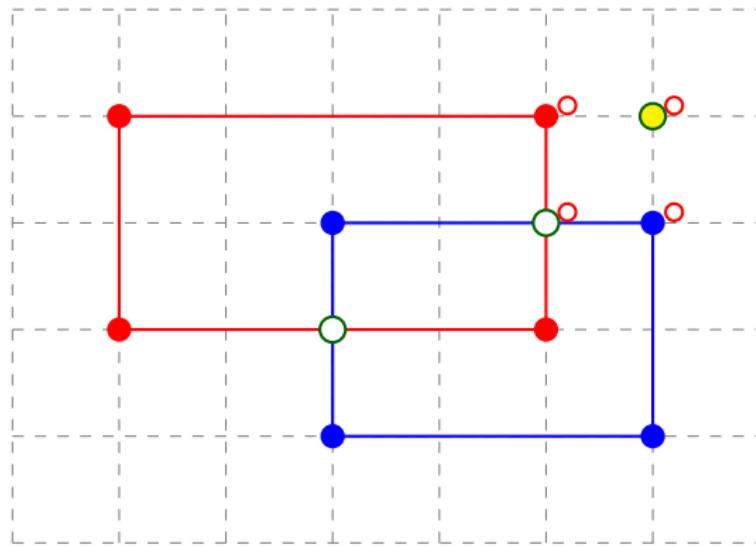
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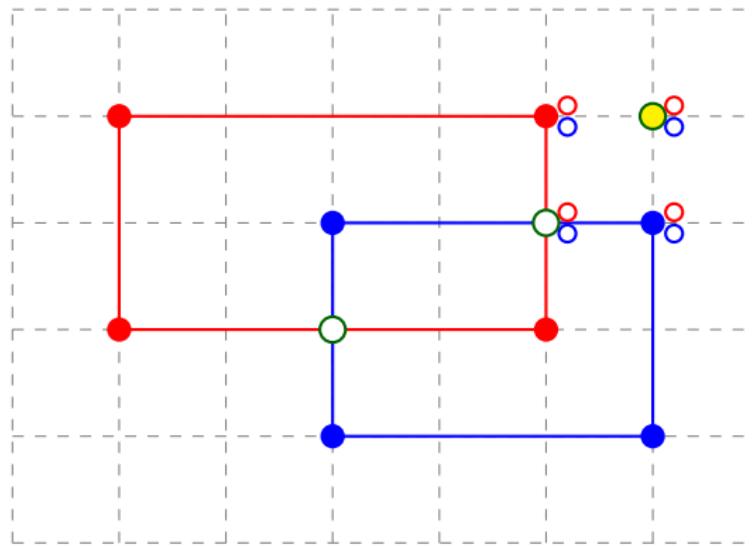
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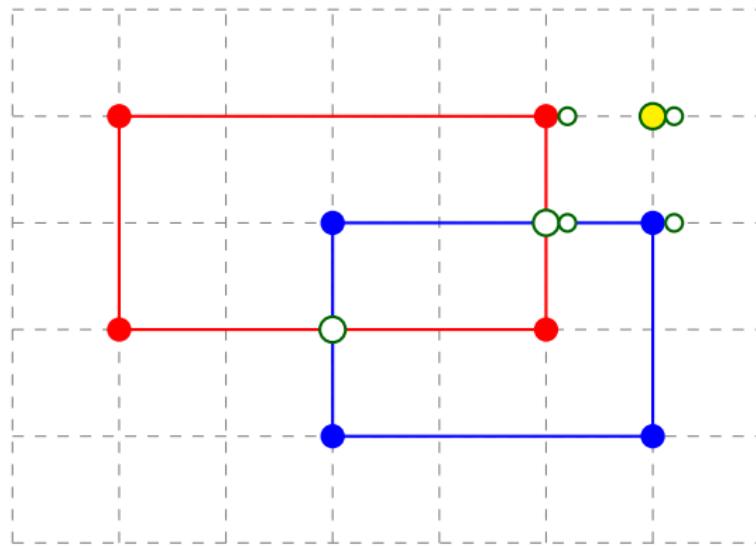
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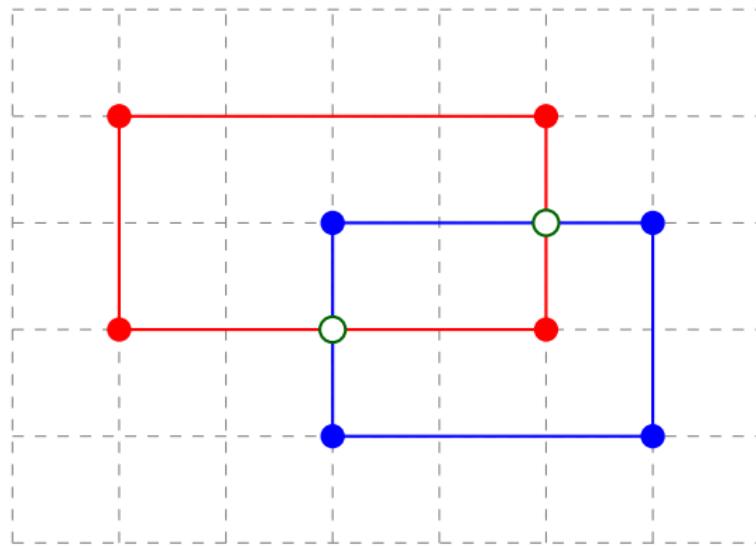
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