Modeling and analysis of hybrid systems
Orthogonal polyhedra

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Informatik 2 - Theory of Hybrid Systems
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- We had a look at state set approximations by convex polyhedra and at basic operations (e.g., testing for membership or intersection) on these.
- Let us now have a look at another representation by orthogonal polyhedra.

Literatur

Oliver Bournez, Oded Maler, and Amir Pnueli:
Orthogonal Polyhedra: Representation and Computation
Hybrid Systems: Computation and Control, LNCS 1569, pp. 46-60, 1999

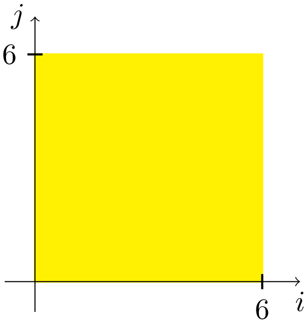
Contents

The real domain

Definition

- Domain: bounded subset $X = [0, m]^d \subseteq \mathbb{R}^d$ $(m \in \mathbb{N}_+)$ of the reals (can be extended to $X = \mathbb{R}^d_+$).
- Elements of X are denoted by $\mathbf{x} = (x_1, \ldots, x_d)$, zero vector $\mathbf{0}$, unit vector $\mathbf{1}$.

$X = [0, 6]^2$



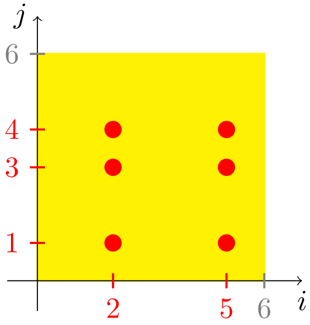
Grids

Definition

A d-dimensional grid associated with $X = [0, m]^d \subseteq \mathbb{R}^d$ ($m \in \mathbb{N}_+$) is a product of d subsets of $\{0, 1, \ldots, m-1\}$.

2-dimensional grid:

$$\{2, 5\} \times \{1, 3, 4\}$$

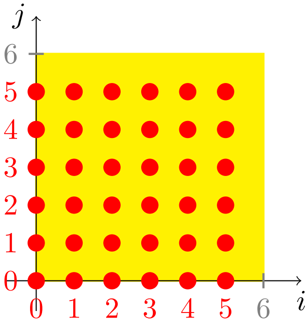


Grids

Definition

The elementary grid associated with $X = [0, m]^d \subseteq \mathbb{R}^d$ $(m \in \mathbb{N}_+)$ is $\mathbf{G} = \{0, 1, \ldots, m-1\}^d \subseteq \mathbb{N}^d$.

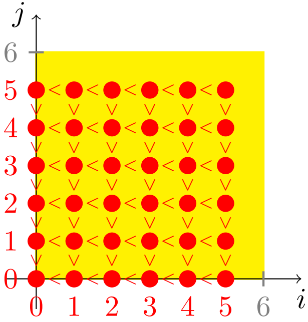
$$G = \{0, \ldots, 5\} \times \{0, \ldots, 5\}$$



Grids

The grid admits a natural partial order with $(m-1,\ldots,m-1)$ on the top and $\mathbf{0}$ as bottom.

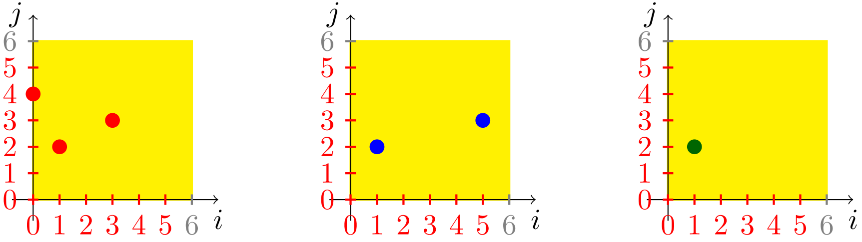
$$G = \{0,\ldots,5\} \times \{0,\ldots,5\}$$



The set of subsets of the elementary grid $\mathbf{G}$ forms a
Boolean algebra $(2^{\mathbf{G}}, \cap, \cup, \sim)$ under the set-theoretic operations

- $A \cup B$
- $A \cap B$
- $\sim A = \mathsf{G} \backslash A$

for $A, B \subseteq \mathbf{G} \subset \mathbb{N}^d$.

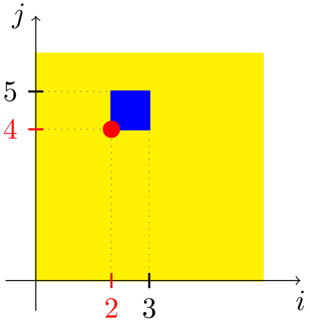


$\{(0,4),(1,2),(3,3)\} \;\cap\; \{(1,2),(5,3)\} \;=\; \{(1,2)\}$

Definition (Elementary box)

- The elementary box associated with a grid point $\mathbf{x} = (x_1, \ldots, x_d)$ is $B(\mathbf{x}) = [x_1, x_1 + 1] \times \ldots, \times [x_d, x_d + 1]$.
- The point $\mathbf{x}$ is called the leftmost corner of $B(\mathbf{x})$.
- The set of elementary boxes is denoted by $\mathbf{B}$.

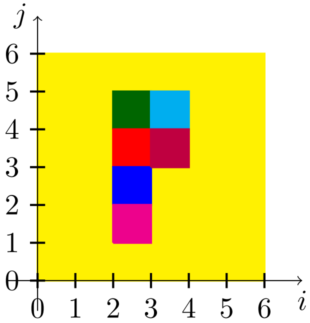
$$B((2,4)) = [2,3] \times [4,5]$$



Definition (Orthogonal polyhedra)

An orthogonal polyhedron P is a union of elementary boxes, i.e., an element of $2^{\mathbf{B}}$.

$$\{B((2,4))\} \cup \{B((3,4))\} \cup$$
$$\{B((2,3))\} \cup \{B((3,3))\} \cup$$
$$\{B((2,2))\} \cup$$
$$\{B((2,1))\}$$



Boolean algebra of orthogonal polyhedra

The set $2^{\mathbf{B}}$ of orthogonal polyhedra is closed under the following operations:

- $A \sqcup B = A \cup B$
- $A \sqcap B = cl(int(A) \cap int(B))$
- $\neg A = cl(\sim A)$

with

- int the interior operator yielding the largest open set $int(A)$ contained in A, and
- cl the topological closure operator yielding the smallest closed set $cl(A)$ containing A.

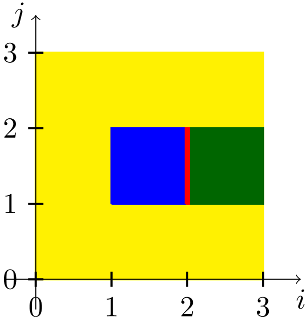
The set of orthogonal polyhedra forms a Boolean algebra $(2^{\mathbf{B}}, \sqcap, \sqcup, \neg)$.

$$A \sqcap B = cl(int(A) \cap int(B))$$

$([1,2] \times [1,2]) \sqcap ([2,3] \times [1,2]) =$

$cl(((1,2) \times (1,2)) \sqcap ((2,3) \times (1,2))) =$

$cl(\emptyset) = \emptyset$



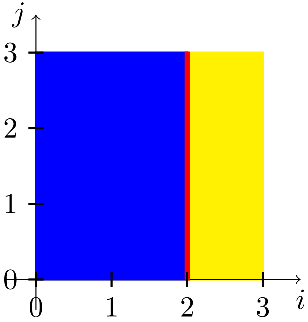
Note: $([1,2] \times [1,2]) \cap ([2,3] \times [1,2]) = [2,2] \times [1,2]$

$$\neg A = cl(\sim A)$$

$\neg([0,2] \times [0,3]) =$

$cl(\sim([0,2] \times [0,3])) =$

$cl((2,3] \times [0,3])) = [2,3] \times [0,3]$

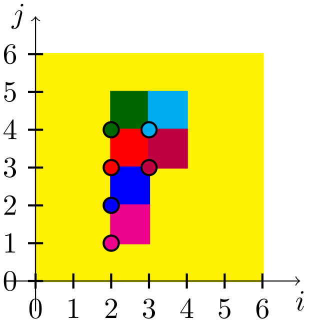


Note: $\sim([0,2] \times [0,3]) = (2,3] \times [0,3]$

Connections

The bijection between $\mathbf{G}$ and $\mathbf{B}$ which associates every elementary box with its leftmost corner generates an isomorphism between $(2^{\mathbf{G}}, \cap, \cup, \sim)$ and $(2^{\mathbf{B}}, \sqcap, \sqcup, \neg)$.

Thus we can switch between point-based and box-based terminology according to what serves better the intuition.



Color function

Definition (Color function)

Let P be an orthogonal polyhedron. The color function $c : X \to \{0,1\}$ is defined by

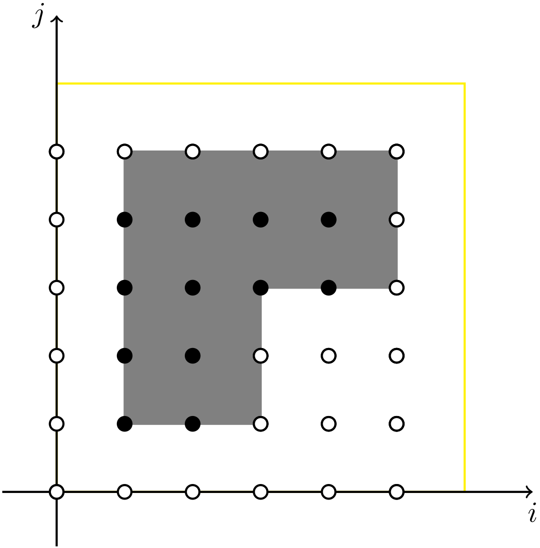
$$c(\mathbf{x}) = \begin{cases} 1 & \textit{if } \mathbf{x} \textit{ is a grid point and } B(\mathbf{x}) \subseteq P \\ 0 & \textit{otherwise} \end{cases}$$

for all $\mathbf{x} \in X$.

- If $c(\mathbf{x}) = 1$ we say that $\mathbf{x}$ is black and that $B(\mathbf{x})$ is full.
- If $c(\mathbf{x}) = 0$ we say that $\mathbf{x}$ is white and that $B(\mathbf{x})$ is empty.

Note that c almost coincides with the characteristic function of P as a subset of X. It differs from it only on right-boundary points.

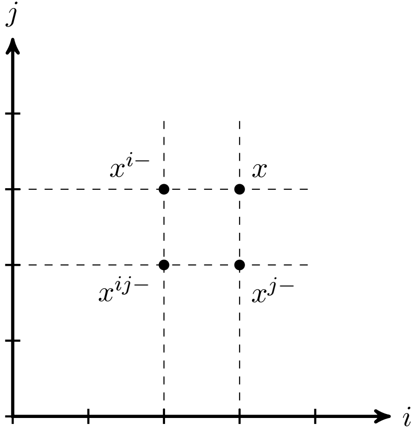
Coloring



The following definitions capture the intuitive meaning of a facet and a vertex and, in particular, that the boundary of an orthogonal polyhedron is the union of its facets.

Definition (i-predecessor)

The i-predecessor of a grid point $\mathbf{x} = (x_1, \dots, x_d) \in X$ is $\mathbf{x}^{i-} = (x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_d)$. We use $\mathbf{x}^{ij-}$ to denote $(\mathbf{x}^{i-})^{j-}$. When $\mathbf{x}$ has no i-predecessor, we write $\perp$ for the predecessor value.



Definition (Neighborhood)

The neighborhood of a grid point $\mathbf{x}$ is the set

$$\mathcal{N}(\mathbf{x}) = \{x_1 - 1, x_1\} \times \ldots \times \{x_d - 1, x_d\}$$

(the vertices of a box lying between $\mathbf{x} - \mathbf{1}$ and $\mathbf{x}$). For every i, $\mathcal{N}(\mathbf{x})$ can be partitioned into left and right i-neighborhoods

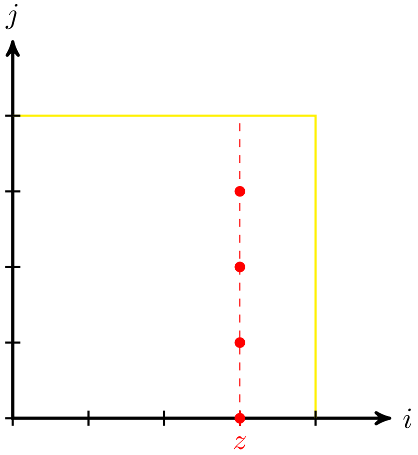
$$\mathcal{N}^{i-}(\mathbf{x}) = \{x_1 - 1, x_1\} \times \ldots \times \{x_i - 1\} \times \{x_d - 1, x_d\}$$

and

$$\mathcal{N}^{i}(\mathbf{x}) = \{x_1 - 1, x_1\} \times \ldots \times \{x_i\} \times \{x_d - 1, x_d\}.$$

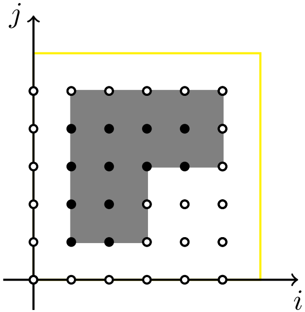
Definition (i-hyperplane)

An i-hyperplane is a $(d-1)$-dimensional subset $H_{i,z}$ of X consisting of all points $\mathbf{x}$ satisfying $x_i = z$.



Observations:

- Facets are $d-1$-dimensional polyhedra.
- As such, facets are subsets of i-hyperplanes.
- The coloring changes on facets.
- White vertices need special care (closure to the “right”).



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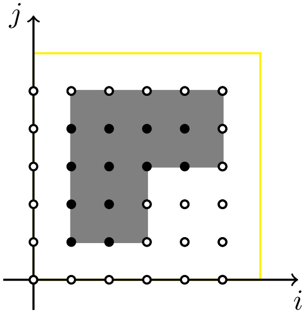


Definition (i-facet)

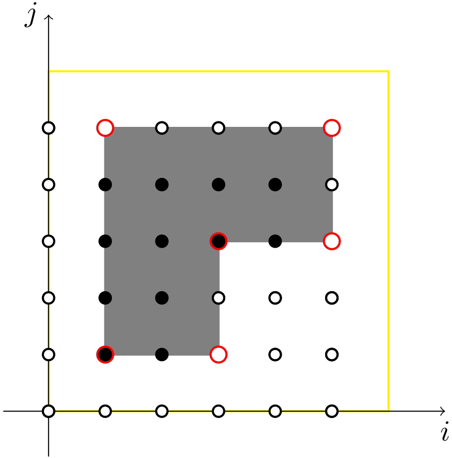
An i-facet of an orthogonal polyhedron P with color function c is

$$F_{i,z}(P) = cl\{\mathbf{x} \in H_{i,z} | c(\mathbf{x}) \neq c(\mathbf{x}^{i-})\}$$

for some integer $z \in [0, m)$.

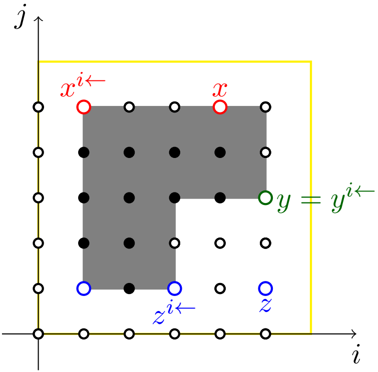
Definition (Vertex)

A vertex is a non-empty intersection of d distinct facets. The set of vertices of an orthogonal polyhedron P is denoted by $V(P)$.



Definition (i-vertex-predecessor)

- An i-vertex-predecessor of $\mathbf{x} = (x_1, \ldots, x_d) \in X$ is a vertex of the form $(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_d)$ for some integer $z \in [0, x_i]$. When $\mathbf{x}$ has no i-vertex-predecessor, we write $\perp$ for its value.
- The first i-vertex-predecessor of $\mathbf{x}$, denoted by $x^{i\leftarrow}$, is the one with the maximal z.



A representation scheme for $2^{\mathbf{B}}$ ($2^{\mathbf{G}}$) is a set $\mathcal{E}$ of syntactic objects such that there is a surjective function ϕ from $\mathcal{E}$ to $2^{\mathbf{B}}$, i.e., every syntactic object represents at most one polyhedron and every polyhedron has at least one corresponding object.

If ϕ is an injection we say that the representation is canonical, i.e., every polyhedron has a unique representation.

Possible representation schemes:

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  - This representation is non-canonical.
- Vertex representation: consists of the set $\{(\mathbf{x}, c(\mathbf{x})) | \mathbf{x}\ is\ a\ vertex\}$, i.e., the vertices of P along with their color.
  - This representation is canonical.
  - The vertices alone is not a representation.
  - Not every set of points and colors is a valid representation of a polyhedron.

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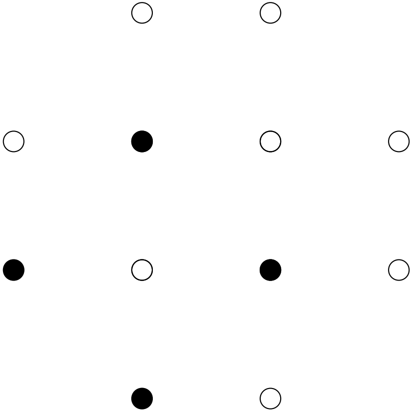
  - The vertices alone is not a representation.
  - Not every set of points and colors is a valid representation of a polyhedron.
- Neighborhood representation: the colors of all the 2^d points in the neighborhoods of the vertices is attached as additional information.
- Extreme vertex representation: instead of maintaining all the neighborhood of each vertex, it suffices to keep only the *parity* of the number of black points in that neighborhood. In fact, it suffices to keep only vertices with odd parity.

Vertex representation



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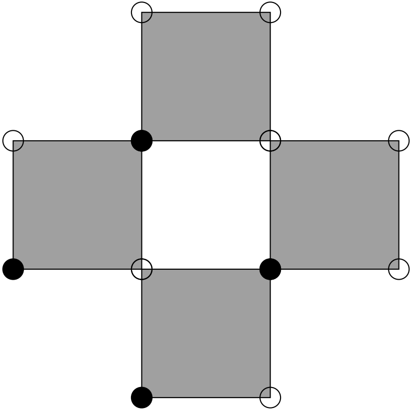

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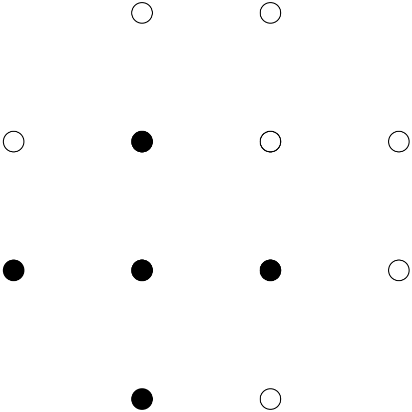
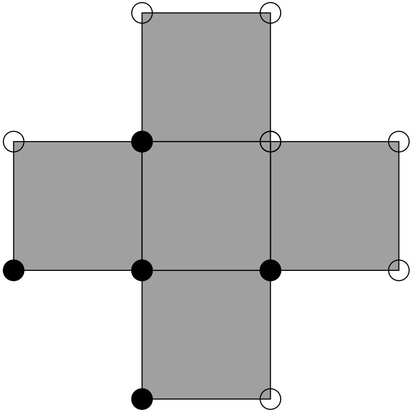
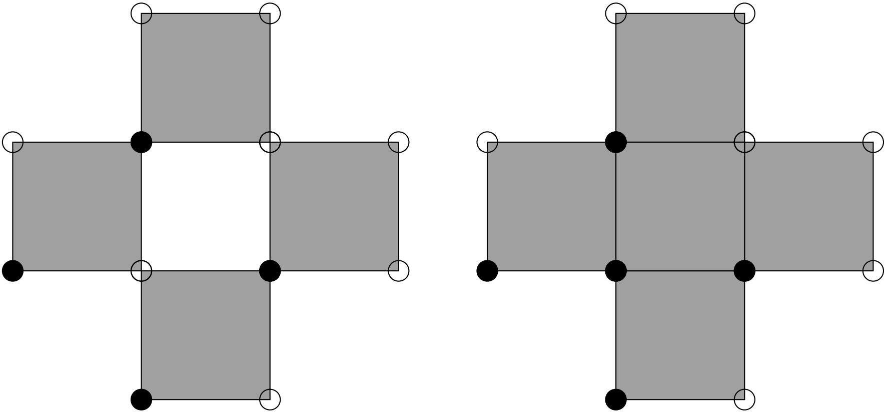


Vertex representation



Vertex representation



Contents

Membership problem

The membership problem

Given a representation of a polyhedron P and a grid point $\mathbf{x}$, determine $c(\mathbf{x})$, that is, whether $B(\mathbf{x}) \subseteq P$.

Contents

Membership problem for the vertex representation

Observations

- A point $\mathbf{x}$ is on an i-facet iff

$$\exists \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}).\ c(\mathbf{x}'^{i-}) \neq c(\mathbf{x}').$$

- A point $\mathbf{x}$ is a vertex iff

$$\forall i \in \{1, \ldots, d\}.\ \exists \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}).\ c(\mathbf{x}'^{i-}) \neq c(\mathbf{x}').$$

- A point $\mathbf{x}$ is not a vertex iff

$$\exists i \in \{1, \ldots, d\}.\ \forall \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}).\ c(\mathbf{x}'^{i-}) = c(\mathbf{x}').$$

Example

For $d = 2$ and $\mathbf{x} = (x_1, x_2)$ it means:

- $\mathbf{x}$ is on a 1-facet iff

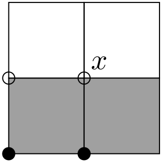
$$c(x_1 - 1, x_2 - 1) \neq c(x_1, x_2 - 1) \vee c(x_1 - 1, x_2) \neq c(x_1, x_2).$$

- $\mathbf{x}$ is on a 2-facet iff

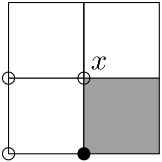
$$c(x_1 - 1, x_2 - 1) \neq c(x_1 - 1, x_2) \vee c(x_1, x_2 - 1) \neq c(x_1, x_2).$$

- $\mathbf{x}$ is a vertex iff both of the above hold.
- $\mathbf{x}$ is not a vertex iff one of the above does not hold.

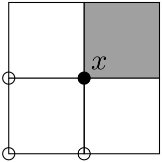
Example



$$c(x_1, x_2 - 1) = c(x_1, x_2) \wedge$$
$$c(x_1 - 1, x_2 - 1) = c(x_1, x_2 - 1)$$



$$c(x_1 - 1, x_2 - 1) \neq c(x_1, x_2 - 1)$$



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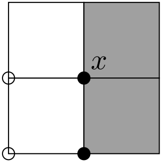

$$c(x_1, x_2 - 1) \neq c(x_1, x_2) \wedge$$
$$c(x_1 - 1, x_2 - 1) \neq c(x_1, x_2 - 1)$$

Color computation

Lemma (Color of a non-vertex)

Let $\mathbf{x}$ *be a non-vertex. Then there exists a direction* $j \in \{1, \ldots, d\}$ *such that*

$$\forall \mathbf{x}' \in \mathcal{N}^j(\mathbf{x}) \backslash \{\mathbf{x}\}.\ c(\mathbf{x}'^{j-}) = c(\mathbf{x}').$$

Let j *be such a direction. Then* $c(\mathbf{x}) = c(\mathbf{x}^{j-})$.

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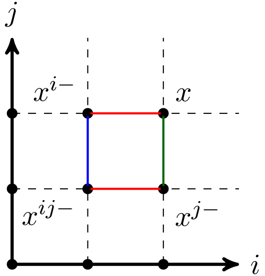
Let j *be such a direction. Then* $c(\mathbf{x}) = c(\mathbf{x}^{j-})$*.*

Proof: A point $\mathbf{x}$ is not a vertex iff

$$\exists i \in \{1, \ldots, d\}.\ \forall \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}).\ c(\mathbf{x}'^{i-}) = c(\mathbf{x}').$$



Thus j always exists. Let i and j be two dimensions satisfying the above requirements.
Case 1: $j = i$: Straightforward
Case 2: $j \neq i$: For i we have $c(\mathbf{x}^{i-}) = c(\mathbf{x})$ and $c(\mathbf{x}^{ij-}) = c(\mathbf{x}^{j-})$. For j we have $c(\mathbf{x}^{ij-}) = c(\mathbf{x}^{i-})$.
Thus $c(\mathbf{x}) = c(\mathbf{x}^{j-})$.

Complexity

Consequently we can calculate the color of a non-vertex $\mathbf{x}$ based on the color of all points in $\mathcal{N}(\mathbf{x}) - \{\mathbf{x}\}$: just find some j satisfying the conditions of the above lemma and let $c(\mathbf{x}) = c(\mathbf{x}^{j-})$.

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- We must recursively determine the color of at most n^d grid points.
- For each of them we must check at most d dimensions if they satisfy the condition of the lemma on the color of a non-vertex.
- Checking the condition invokes $2^d - 1$ color comparisions.

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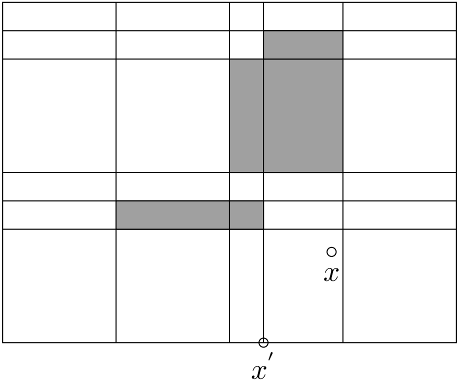
- For each of them we must check at most d dimensions if they satisfy the condition of the lemma on the color of a non-vertex.
- Checking the condition invokes $2^d - 1$ color comparisions.

However, this algorithm is not very efficient, because in the worst-case one has to calculate the color of all the grid points between $\mathbf{0}$ and $\mathbf{x}$.

Induced grid

We can improve it using the notion of an induced grid: let the i-scale of P be the set of the i-coordinates of the vertices of P, and let the induced grid be the Cartesian product of its i-scales.



Induced grid

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- The induced grid is the smallest (coarsest) grid containing all the vertices.
- Every rectangle in the induced grid has a uniform color.
- Calculating the color of a point reduces to finding its closest "dominating" point on the induced grid and applying the algorithm to that grid in $\mathcal{O}(n^d d 2^d)$ time.

Contents

Membership problem for the neighborhood representation

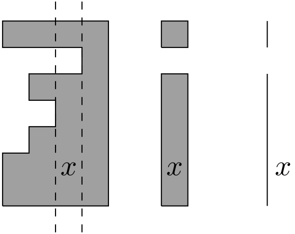
We introduce an $\mathcal{O}(n \log n)$ membership algorithm for the neighborhood representation, based on successive projections of P into polyhedra of smaller dimension.

Membership problem for the neighborhood representation

Definition (i-slice and i-section)

Let P be an orthogonal polyhedron and z an integer in $[0, m)$.

- The i-slice of P at z is the d-dimensional orthogonal polyhedron $J_{i,z}(P) = P \sqcap \{\mathbf{x} | z \leq x_i \leq z + 1\}$.
- The i-section of P at z is the $(d-1)$-dimensional orthogonal polyhedron $\mathcal{J}_{i,z}(P) = J_{i,z}(P) \cap H_{i,z}$.



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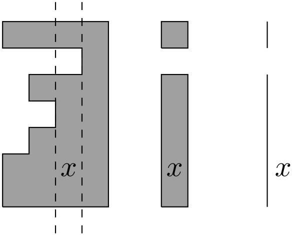

Clearly, the membership of $\mathbf{x} = (x_1, \ldots, x_d)$ can be reduced into membership in $\mathcal{J}_{i,x_i}(P)$, which is a $(d-1)$-dimensional problem. By successively reducing dimensionality for every i we obtain a point whose color is that of $\mathbf{x}$.

Calculating the i-sections for the neighborhood representation

How can the main computational activity, the calculation of i-sections, be done using the neighborhood representation?

Lemma (Vertex of a section)

Let P be an orthogonal polyhedron and let P' be its i-section at $x_i = z$. A point $\mathbf{x}$ is a vertex of P' iff $\mathbf{y} = \mathbf{x}^{i\leftarrow} \neq \bot$ and for every $j \neq i$ there exists $\mathbf{x}' \in \mathcal{N}^i(\mathbf{y}) \cap \mathcal{N}^j(\mathbf{y})$ such that $c(\mathbf{x}'^{j-}) \neq c(\mathbf{x}')$.

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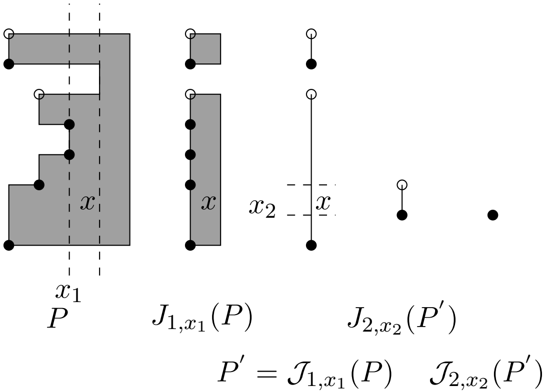

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- Hence it takes $\mathcal{O}(nd(\log n + 2^d))$ to get rid of one dimension.
- This is repeated d times until p is contracted into a point.

Contents

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- We use $parity(\mathbf{x})$ to denote the parity of the number of black points in $\mathcal{N}(\mathbf{x})$.
- A point $\mathbf{x}$ is said to be extreme if $parity(\mathbf{x}) = 1$.

Membership problem for the extreme vertex representation

Lemma

An extreme point is a vertex.

Proof: By induction on the dimension d. The base case $d = 1$ is immediate. For $d > 1$, choose $i \in \{1, \ldots, d\}$. Exactly one of $\mathcal{N}^{i-}(\mathbf{x})$ and $\mathcal{N}^{i}(\mathbf{x})$ contains an odd number of black points. Assume w.l.o.g. that it is $\mathcal{N}^{i}(\mathbf{x})$. By induction hypothesis $\mathbf{x}$ is a vertex in $\mathcal{J}_{i,x_i}(P)$. I.e., for every $j \neq i$ there exists $\mathbf{x}' \in \mathcal{N}^{j}(\mathbf{x})$ such that $c(x'^{j-}) \neq c(x')$. Since one cannot have $c(\mathbf{x}') = c(\mathbf{x}'^{i-})$ for all $\mathbf{x}' \in \mathcal{N}^{i}(\mathbf{x})$, $\mathbf{x}$ is a vertex of P.

The converse is not true, i.e., vertices need not be extreme.

- An extreme vertex representation consists in representing an orthogonal polyhedron by the set of its extreme vertices. (Additionally, the color of the origin is stored in a bit. From this information the colors of all extreme vertices can be inferred.)

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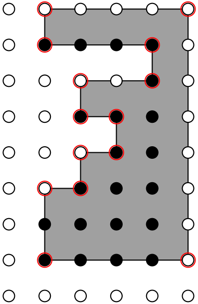

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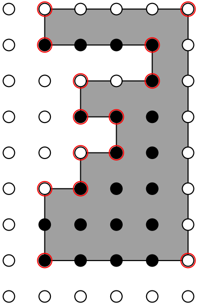

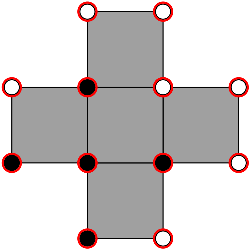

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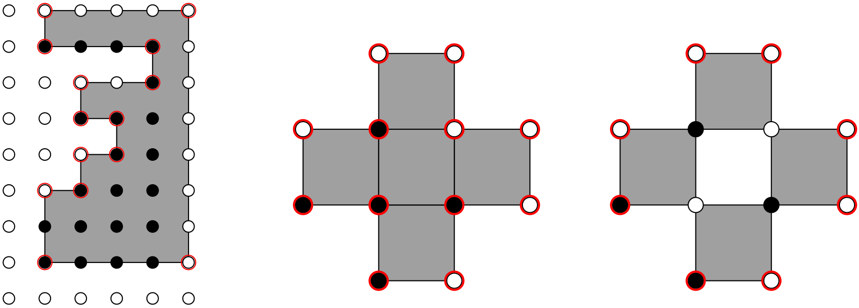

The membership problem is solved again by projection. For that we need again a rule to determine which points of an i-section are extreme vertices.

Lemma (Extreme vertices of a section)

Let P be an orthogonal polyhedron and let $P' = \mathcal{J}_{i,z}(P)$. A point $\mathbf{x}$ is an extreme vertex of P' iff it has an odd number of extreme i-vertex-predecessors.

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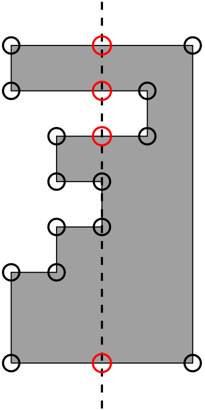

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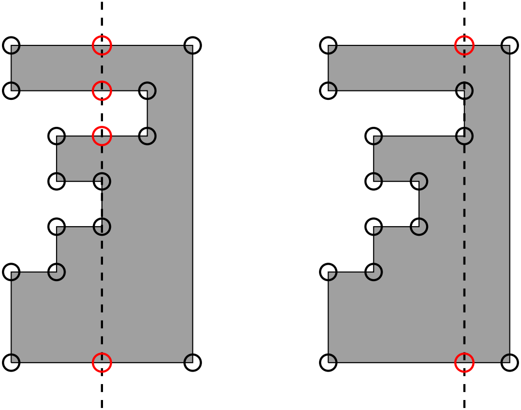

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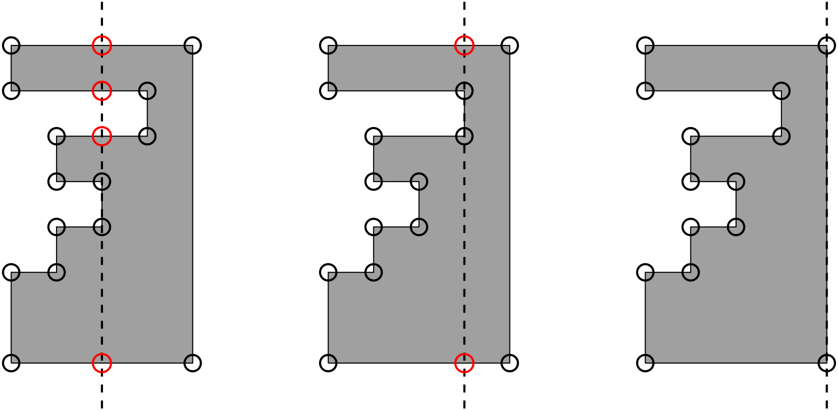

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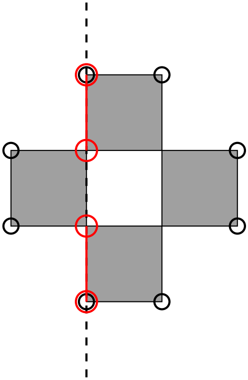

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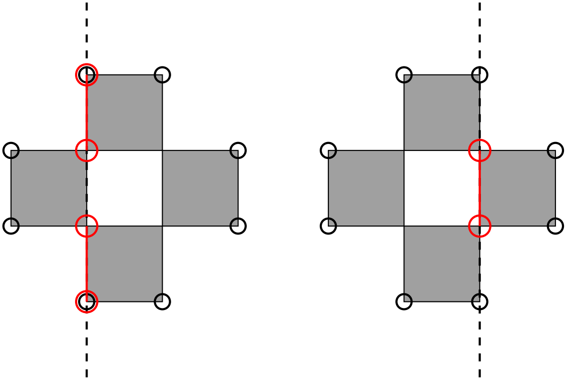

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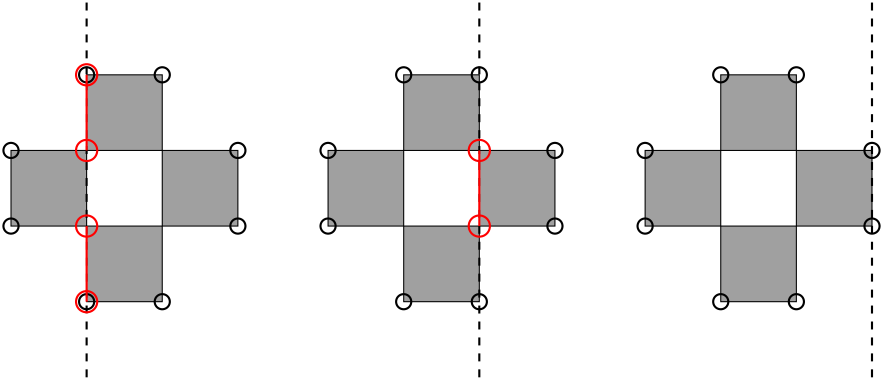
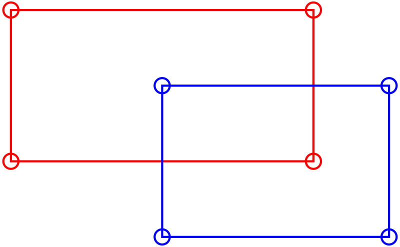


Contents

Intersection

We assume two polyhedra P_1 and P_2 with n_1 and n_2 vertices, respectively. After intersection some vertices disappear and some new vertices are created.



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Lemma

A point $\mathbf{x}$ is a vertex of $P_1 \cap P_2$ only if for every dimension i, $\mathbf{x}$ is on an i-facet of P_1 or on an i-facet of P_2.

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Lemma

Let $\mathbf{x}$ *be a vertex of* $P_1 \cap P_2$ *which is not an original vertex.*
Then there exists a vertex $\mathbf{y}_1$ *of* P_1 *and a vertex* $\mathbf{y}_2$ *of* P_2 *such that* $\mathbf{x} = \max(\mathbf{y}_1, \mathbf{y}_2)$*, where* $\max$ *is applied componentwise.*

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Conclusion: the candidates for being vertices of $P_1 \cap P_2$ are restricted to:

$$V(P_1) \cup V(P_2) \cup \{\mathbf{x} | \exists \mathbf{y}_1 \in V(P_1).\ \exists \mathbf{y}_2 \in V(P_2).\ \mathbf{x} = \max(\mathbf{y}_1, \mathbf{y}_2)\}$$

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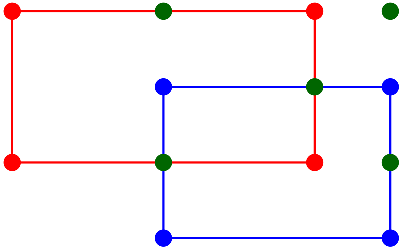
whose number is not greater then $n_1 + n_2 + n_1 n_2$.

Intersection



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  - Use the vertex rules to determine, whether the point is a vertex of the intersection.

Intersection example: Vertex representation

Vertex rule: A point $\mathbf{x}$ is a vertex iff

$$\forall i \in \{1, \ldots, d\}.\ \exists \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}).\ c(\mathbf{x}'^{i-}) \neq c(\mathbf{x}').$$

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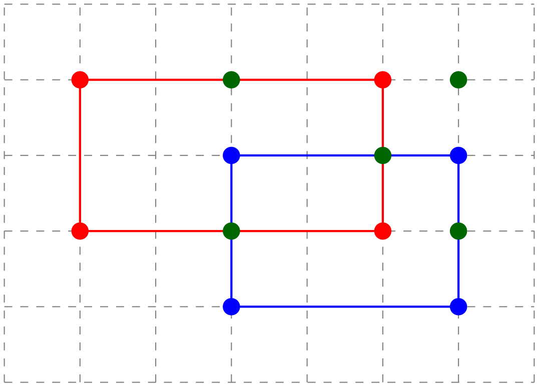

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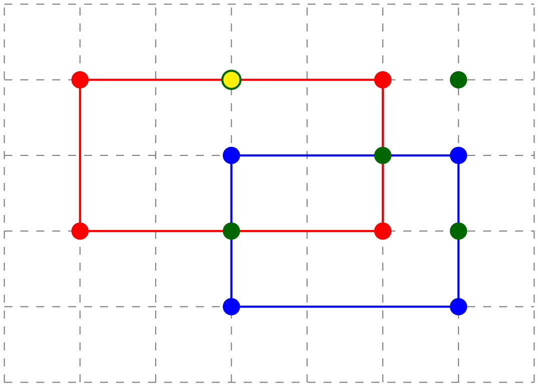

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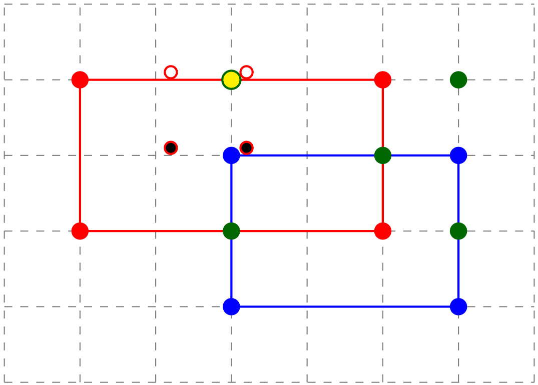

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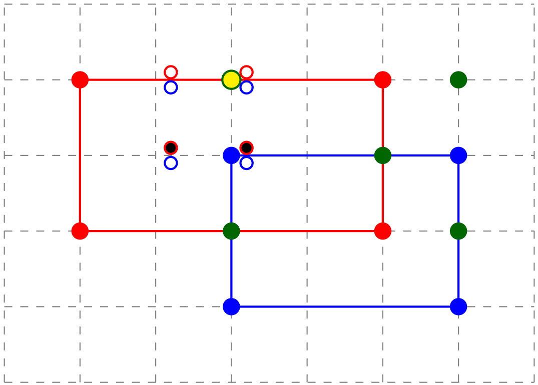

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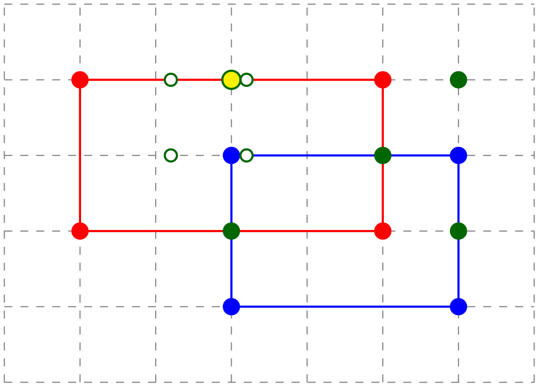

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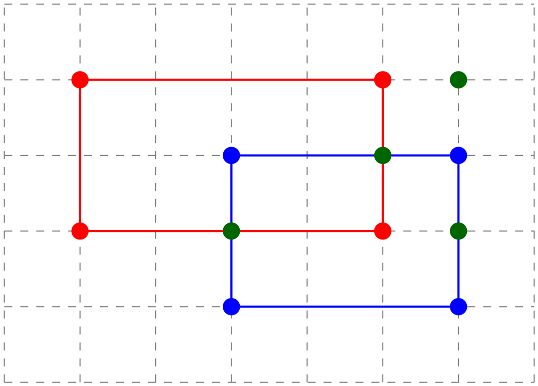

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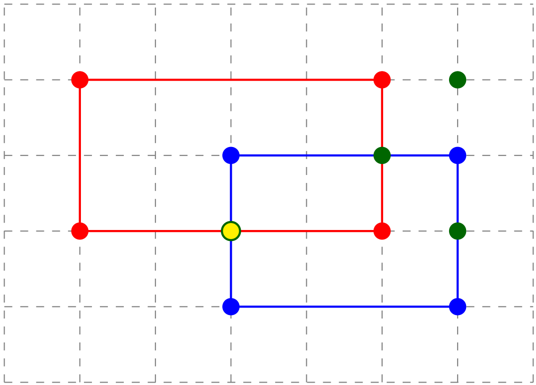

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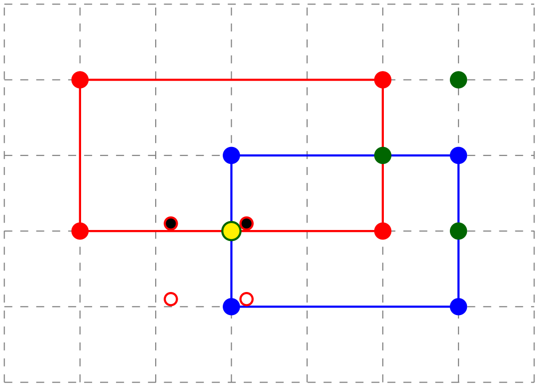

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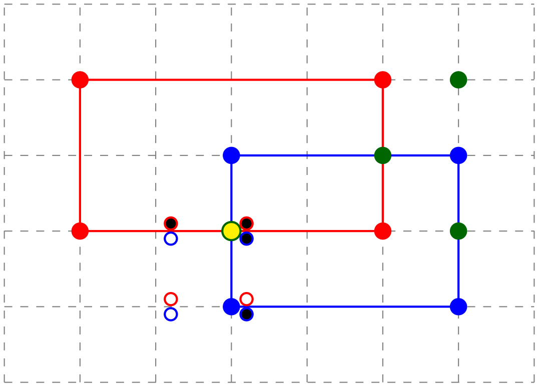

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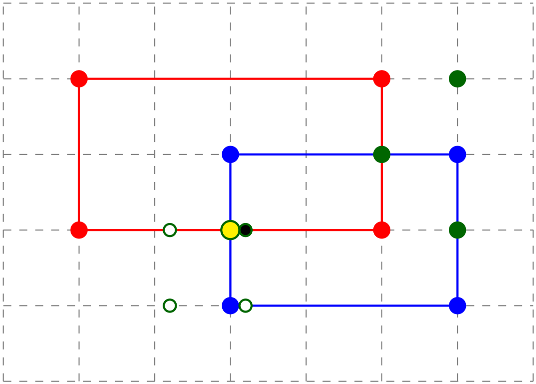

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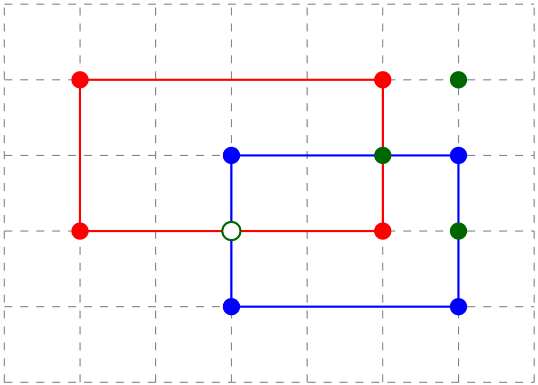
Vertex rule: A point $\mathbf{x}$ is a vertex iff

$$\forall i \in \{1, \ldots, d\}.\ \exists \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}).\ c(\mathbf{x}'^{i-}) \neq c(\mathbf{x}').$$



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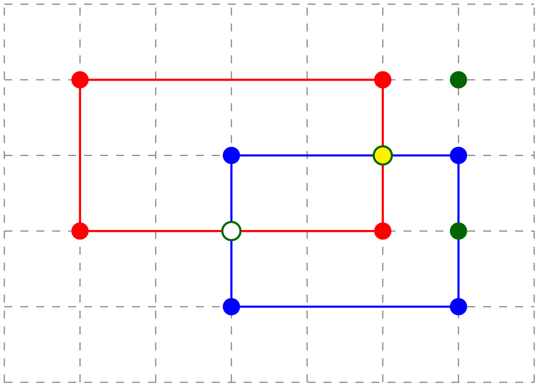

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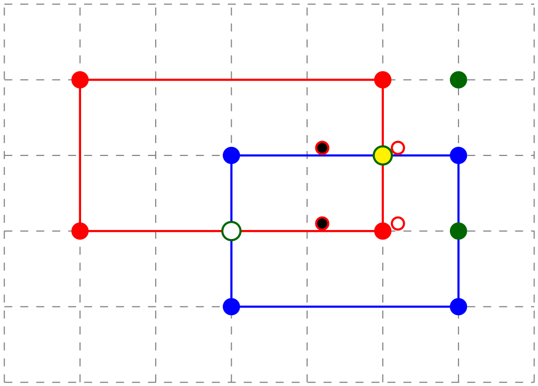

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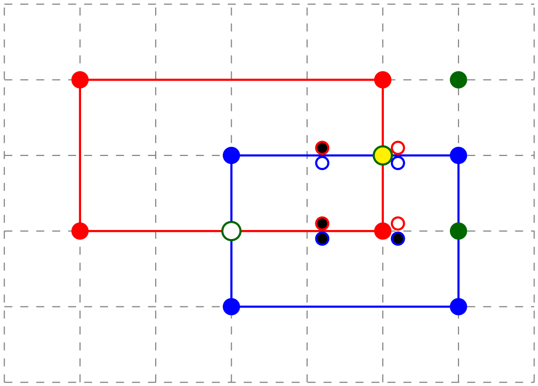

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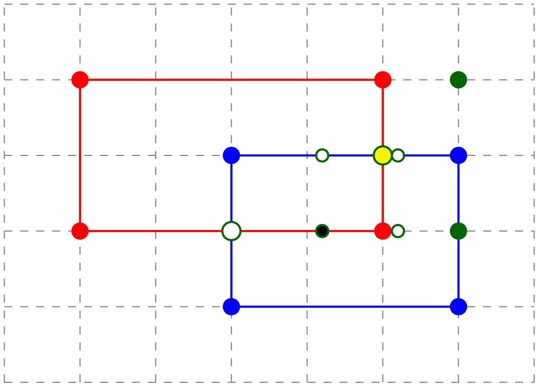

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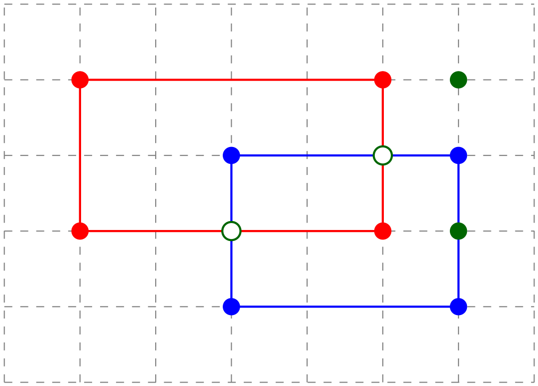

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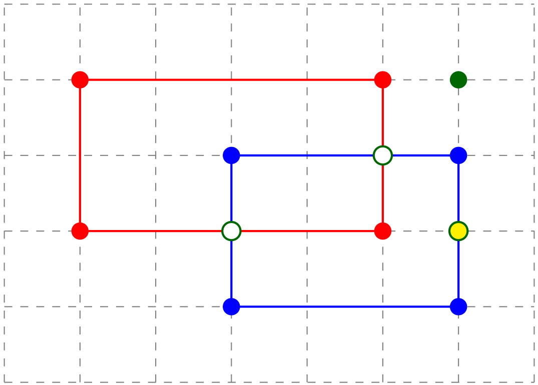

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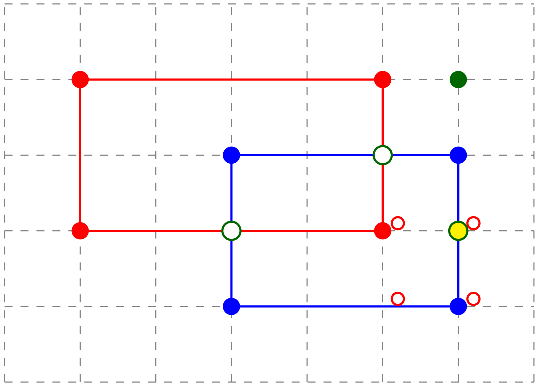

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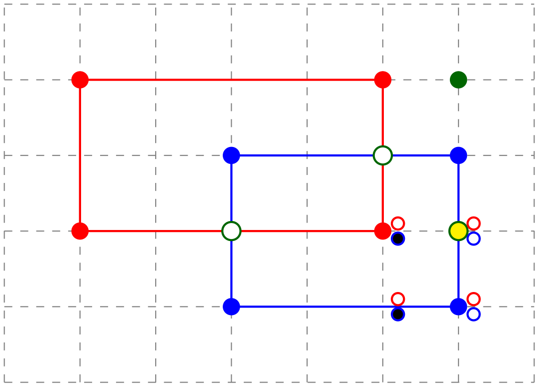

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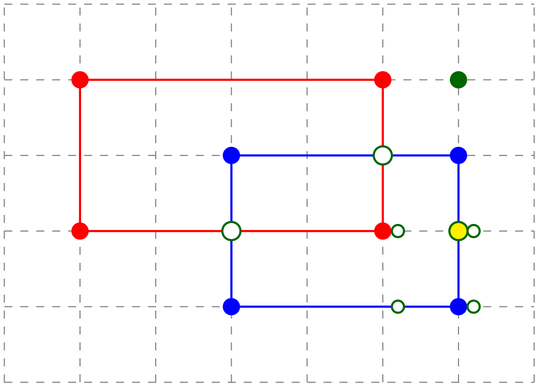

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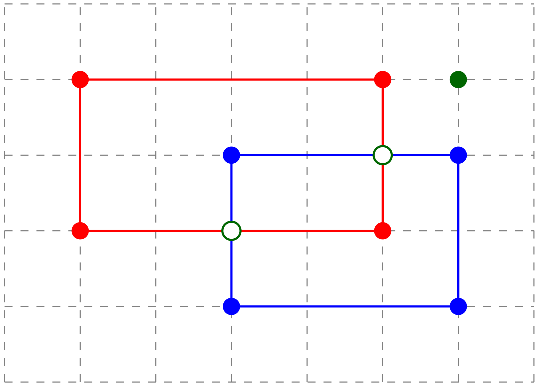

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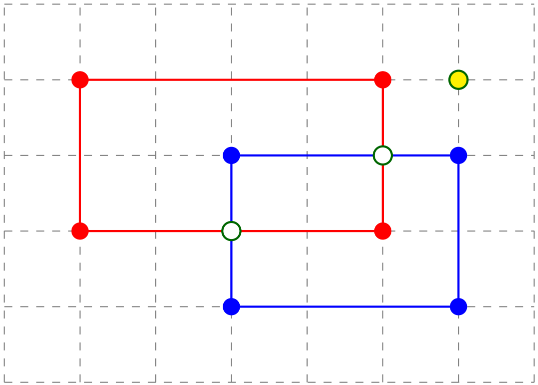

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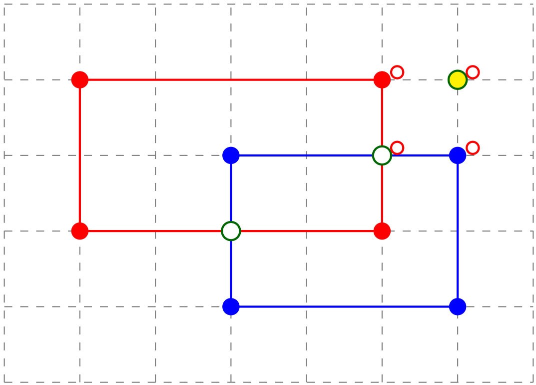

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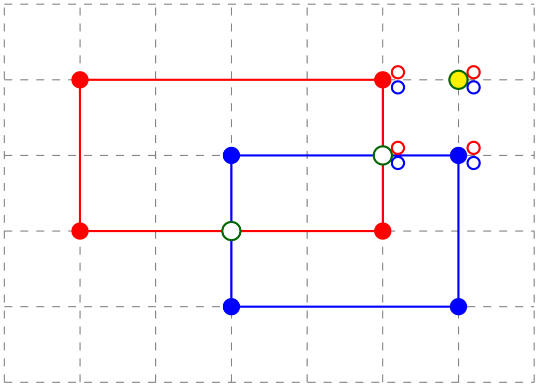

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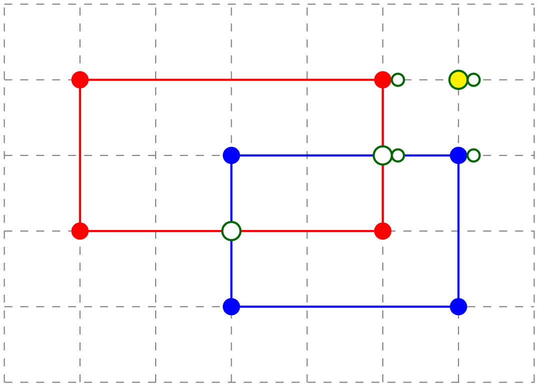

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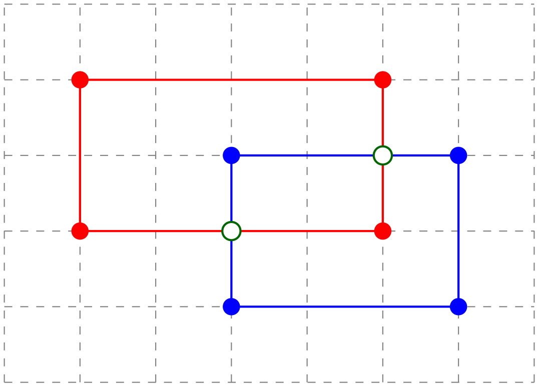

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