Modeling and analysis of hybrid systems

Oriented rectangular hulls

Prof. Dr. Erika Ábrahám

Informatik 2 - Theory of Hybrid Systems
RWTH Aachen

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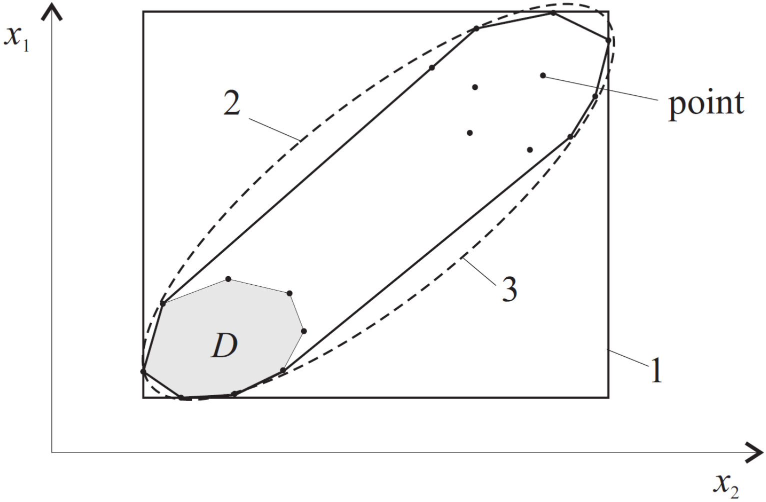
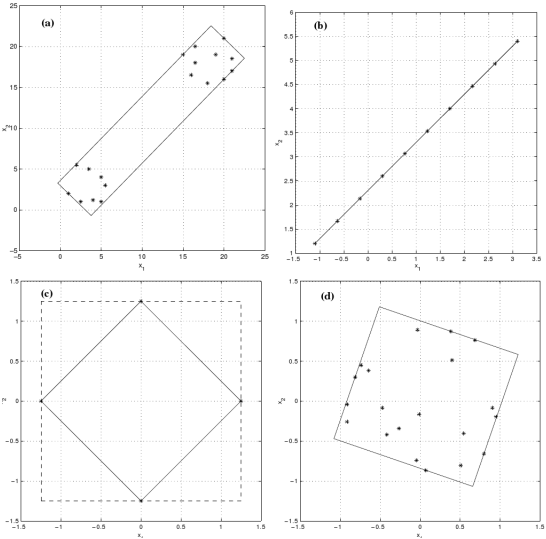
Literatur

Olaf Stursberg and Bruce H. Krogh:
Efficient Representation and Computation of Reachable Sets for Hybrid Systems
Hybrid Systems: Computation and Control, LNCS 2623, pp. 482-497, 2003

Motivation



Oriented rectangular hull



Principal component analysis (PCA)

- transforms some given data
- to a new coordinate system such that
- the greatest variance by any projection of the data comes to lie on the first coordinate (called the first principal component),
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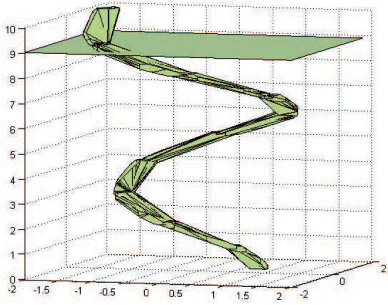
PCA involves the calculation of the eigenvalue decomposition of a data covariance matrix (or singular value decomposition of a data matrix), after mean centering the data for each attribute.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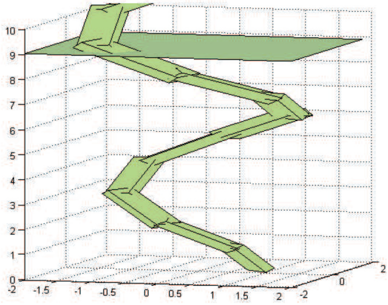
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Given a vector of sample points $X = (x^1, \ldots, x^p)$ with $x^i \in \mathbb{R}^n$, its arithmetic mean is

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We translate the samples such that their arithmetic mean becomes 0:

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In matrix form:

$$\bar{X} = (x^1, \ldots, x^p) = \begin{pmatrix} \bar{x}^1_1 & \cdot & \cdot & \cdot & \bar{x}^p_1 \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \bar{x}^1_n & \cdot & \cdot & \cdot & \bar{x}^p_n \end{pmatrix}.$$

Example

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$$\bar{X} = \begin{pmatrix} -2 & -2 & 0 & 0 & 0 & 2 & 0 & 2 \\ -1.5 & 0.5 & -1.5 & 0.5 & -0.5 & -0.5 & 1.5 & 1.5 \end{pmatrix}$$

For

$$\bar{X} = (x^1, \ldots, x^p) = \begin{pmatrix} \bar{x}_1^1 & \cdot & \cdot & \cdot & \bar{x}_1^p \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \bar{x}_n^1 & \cdot & \cdot & \cdot & \bar{x}_n^p \end{pmatrix}$$

we define the sample covariance matrix

$$\mathrm{Cov}(\bar{X}) = \begin{pmatrix} \mathrm{Cov}(\bar{x}_1, \bar{x}_1) & \cdot & \cdot & \cdot & \mathrm{Cov}(\bar{x}_1, \bar{x}_n) \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \mathrm{Cov}(\bar{x}_n, \bar{x}_1) & \cdot & \cdot & \cdot & \mathrm{Cov}(\bar{x}_n, \bar{x}_n) \end{pmatrix}$$

with

$$\mathrm{Cov}(\bar{x}_i, \bar{x}_j) = \frac{1}{p-1} \sum_{k=1}^{p} \bar{x}_i^k \cdot \bar{x}_j^k$$

for all $0 \leq i, j \leq n$.

Example

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- Cov$(\bar{x}_1, \bar{x}_1) = \frac{1}{7} \sum_{k=1}^{8} \bar{x}_1^k \cdot \bar{x}_1^k = \frac{1}{7}(4+4+4+4) = \frac{16}{7}$

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- $\mathsf{Cov}(\bar{x}_1, \bar{x}_2) = \mathsf{Cov}(\bar{x}_2, \bar{x}_1) = \frac{1}{7} \sum_{k=1}^{8} \bar{x}_1^k \cdot \bar{x}_2^k = \frac{1}{7}(3-1-1+3) = \frac{4}{7}$

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  $\frac{1}{7}((-1.5)^2+0.5^2+(-1.5)^2+0.5^2+(-0.5)^2+(-0.5)^2+1.5^2+1.5^2) = \frac{10}{7}$

Example

$$\mathsf{Cov}(\bar{X}) = \begin{pmatrix} \frac{16}{7} & \frac{4}{7} \\ \frac{4}{7} & \frac{10}{7} \end{pmatrix}$$

Eigenvector and eigenvalue

Given a square matrix A, an eigenvalue λ and its associated eigenvector $\mathbf{v}$ are, by definition, a pair obeying the relation

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Equivalently,

$$(A - \lambda I)\mathbf{v} = 0$$

where I is the identity matrix, implying

$$\mathrm{d}et(A - \lambda I) = 0.$$

Principal component analysis

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- The second principal component corresponds to the same concept after all correlation with the first principal component has been subtracted out from the points.
- Thus, the sum of all the eigenvalues is equal to the sum squared distance of the points with their mean. PCA essentially rotates the set of points around their mean in order to align with the first few principal components. This moves as much of the variance as possible (using a linear transformation) into the first few dimensions.

Example

$$\mathsf{Cov}(\bar{X}) = \begin{pmatrix} \frac{16}{7} & \frac{4}{7} \\ \frac{4}{7} & \frac{10}{7} \end{pmatrix}$$

Eigenvalue computation for 2×2 matrices

The eigenvalues of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be obtained by the characteristic polynomial

$$\mathrm{d}et \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc)$$

with solutions

$$\lambda = \frac{a+d}{2} \pm \sqrt{\frac{(a+d)^2}{4} + bc - ad} = \frac{a+d}{2} \pm \frac{\sqrt{4bc + (a-d)^2}}{2}.$$

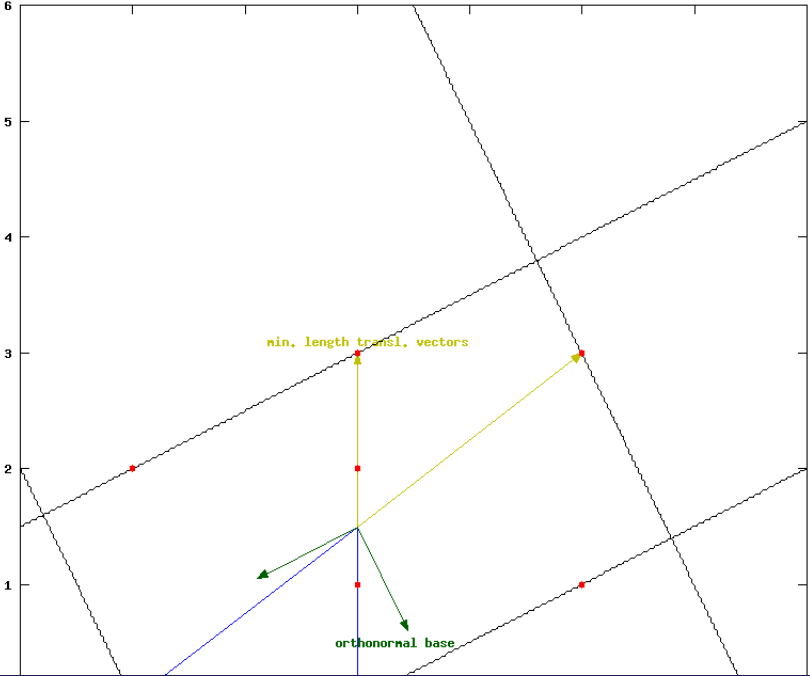
Example

$$\mathsf{Cov}(\bar{X}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{16}{7} & \frac{4}{7} \\ \frac{4}{7} & \frac{10}{7} \end{pmatrix}$$

$$\lambda = \frac{a+d}{2} \pm \frac{\sqrt{4bc + (a-d)^2}}{2} = \frac{13}{7} \pm \frac{5}{7}$$

$$\lambda_1 = \frac{18}{7}$$

$$\lambda_2 = \frac{8}{7}$$


min. length transl. vectors
orthonormal base
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