

# Modeling and analysis of hybrid systems

## Approximation of reachable state sets

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Informatik 2 - Theory of Hybrid Systems  
RWTH Aachen

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Alongkrit Chutinan and Bruce H. Krogh:

Computing Polyhedral Approximations to Flow Pipes for Dynamic Systems

In Proceedings of the 37rd IEEE Conference on Decision and Control, 1998

Olaf Stursberg and Bruce H. Krogh:

Efficient Representation and Computation of Reachable Sets for Hybrid Systems

Hybrid Systems: Computation and Control, LNCS 2623, pp. 482-497, 2003

We had a look at state set approximations by

- convex polyhedra,
- orthogonal polyhedra, and
- oriented rectangular hull,

and at the basic operations

- testing for membership,
- intersection, and
- union

on these.

Thus we can

- approximate state sets and
- compute with them.

How is all this used in the reachability analysis procedure?

# General reachability procedure

Input: Set **Init** of initial states.

Algorithm:

```
 $R^{\text{new}} := \text{Init};$   
 $R := \emptyset;$   
while ( $R^{\text{new}} \neq \emptyset$ ) {  
     $R := R \cup R^{\text{new}};$   
     $R^{\text{new}} := \boxed{\text{Reach}}(R^{\text{new}} \setminus R);$   
}
```

Output: Set **R** of reachable states.

**What is “Reach”?**

# What is “Reach”?

For **hybrid systems**, independently of the exact definition of “Reach”, it will involve the following computations:

Given a state set  $R$ , compute

- the set of states reachable from  $R$  by a **flow** (i.e., time transisiton), and
- the set of states reachable from  $R$  by a **jump** (i.e., discrete transition).

Computing the jump successors, i.e., the flow pipe, of a set can be done with the operations we already introduced.

**The harder part is computing the flow successors.** So let's start with that...

# Approximating a flow pipe

Consider a dynamical system with **state equation**

$$\dot{x} = f(x(t)).$$

We assume  $f$  to be **Lipschitz continuous** so that for every initial state  $x_0$  there is a unique solution  $x(t, x_0)$  to the state equation.

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The set of **reachable states at time  $t$**  from a set of initial states  $X_0$  is defined as

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We describe a solution which approximates the flow pipe by a sequence of **convex polytopes**.

## Definition (Convex polytope)

Let  $POLY(C, d)$  denote the convex polytope defined by the pair  $(C, d) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m$  according to

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- Given a finite set of points  $\Gamma$ , the **convex hull  $CH(\Gamma)$**  of  $\Gamma$  is the smallest convex set that contains  $\Gamma$ .

## Problem statement for polyhedral approximation of flow pipes

Given

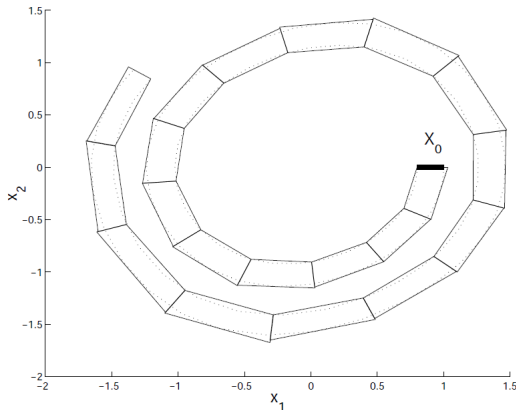
- a set  $X_0$  of initial states which is a polytope, and
- a final time  $t_f$ ,

compute a polyhedral approximation  $\hat{\mathcal{R}}_{[0,t_f]}(X_0)$  to the flow pipe  $\mathcal{R}_{[0,t_f]}(X_0)$  such that

$$\mathcal{R}_{[0,t_f]}(X_0) \subseteq \hat{\mathcal{R}}_{[0,t_f]}(X_0).$$

# Flow pipe segmentation

Since a single convex polyhedra would strongly overapproximate the flow pipe, we compute a **sequence of convex polyhedra**, each approximating a **flow pipe segment**.



## Segmented flow pipe approximation

Let the time interval  $[0, t_f]$  be divided into  $0 < N \in \mathbb{N}$  time segments

$$[0, t_1], [t_1, t_2], \dots, [t_{N-1}, t_f]$$

with  $t_i = i \cdot \frac{t_f}{N}$ .



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We generate an approximation  $\hat{\mathcal{R}}_{[t_1, t_2]}(X_0)$  for each flow pipe segment:

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$$\mathcal{R}_{[t_1, t_2]}(X_0) \subseteq \hat{\mathcal{R}}_{[t_1, t_2]}(X_0).$$

The complete **flow pipe approximation** is the union of the approximation of all  $N$  pipe segments:

$$\mathcal{R}_{[0, t_f]}(X_0) \subseteq \hat{\mathcal{R}}_{[0, t_f]}(X_0) = \bigcup_{k=1, \dots, N} \hat{\mathcal{R}}_{[t_{k-1}, t_k]}(X_0)$$

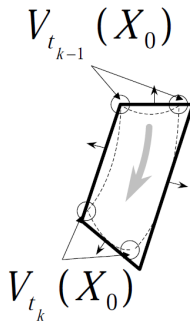
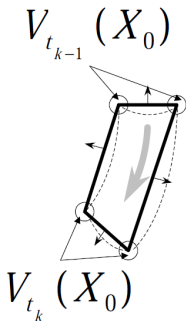
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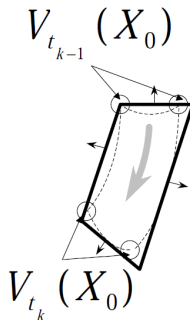
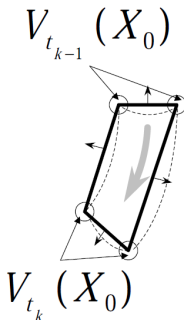
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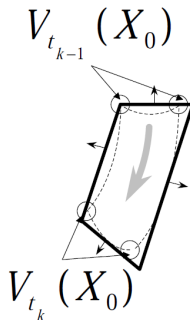
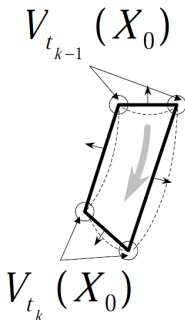
- **Evolve vertices:** Compute the set of points reachable from the vertices of  $X_0$  in time  $t_{i-1}$  and in time  $t_i$ .
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- **Determine hull:** Compute the convex hull of those points.
- **Bloat hull:** Enlarge the hull until it contains all points of the flow pipe segment.



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In particular, we compute the sets  $V_{t_{k-1}}(X_0)$  and  $V_{t_k}(X_0)$  where

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Each point in the above sets can be obtained

- by analytic solution of the state equation and computing the value, or
- by simulation.

## 2. Determine hull

We use the evolved vertices in  $V_{t_{k-1}}(X_0)$  and  $V_{t_k}(X_0)$  to form a **convex hull** which serves as an **initial approximation** to the flow pipe segment  $\mathcal{R}_{[t_{k-1}, t_k]}(X_0)$ , denoted by

$$\Phi_{[t_{k-1}, t_k]}(X_0) = CH(V_{t_{k-1}}(X_0) \cup V_{t_k}(X_0)).$$

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Let  $(C_\Phi, d_\Phi)$  be the matrix-vector pair defining the convex hull, i.e.,

$$\Phi_{[t_{k-1}, t_k]}(X_0) = POLY(C_\Phi, d_\Phi).$$

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- We use the normal vectors to the faces of this convex hull as a set of direction vectors to bloat the convex set until it contains the whole flow pipe segment.
- Given:  $POLY(C_\Phi, d_\Phi)$ .
- We want:  $\mathcal{R}_{[t_{k-1}, t_k]}(X_0) \subseteq POLY(C_\Phi, \boxed{d})$ .



### 3. Bloat hull

- We compute  $d$  as the solution to the following **optimization problem**:

$$\begin{array}{ll} \min_d & \text{volume}[POLY(C_\Phi, d)] \\ \text{s.t.} & \mathcal{R}_{[t_{k-1}, t_k]}(X_0) \subseteq POLY(C_\Phi, d). \end{array} \quad (1)$$

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- The  $i$ th component  $d_i^*$  of the optimum  $d^*$  can be found by solving

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- or, equivalently,

$$\max_{x_0, t} c_i^T x(t, x_0) \quad \text{s.t. } x_0 \in X_0, t \in [t_{k-1}, t_k]. \tag{3}$$

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- **Solution**  $(x_0^*, t^*)$  to 3  $\rightarrow$

**Solution**  $x(t^*, x_0^*)$  to 2  $\rightarrow$

**Solution**  $d_i^* = c_i^T x(t^*, x_0^*)$  to 1.

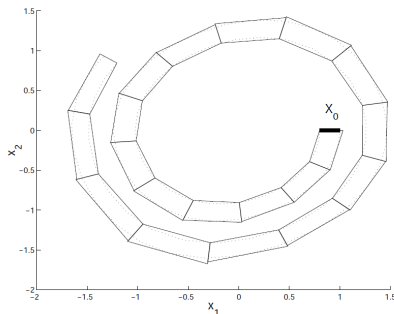
# Example

- Van der Pol equation:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -0.2(x_1^2 - 1)x_2 - x_1.$$

- Initial set:  $X_0 = \{(x_1, x_2) \mid 0.8 \leq x_1 \leq 1 \wedge x_2 = 0\}$ .
- Time:  $t_f = 10$ .
- Segments: 20

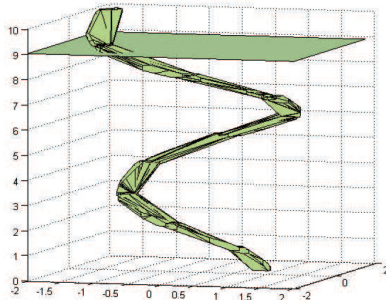


# Other geometries for approximation

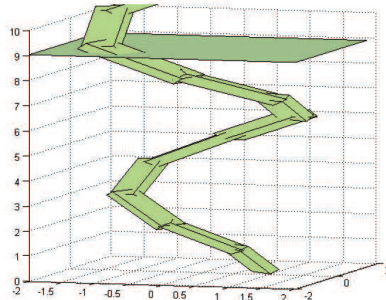
- Van der Pol equation with a third variable being a clock.

- Approximation

with convex polyhedra and



with oriented rectangular hull:



# Partitioning the initial set

Van der Pol system with initial set  $X_0 = \{(x_1, x_2) \mid 5 \leq x_1 \leq 45 \wedge x_2 = 0\}$ .

