# Reachability Probabilities in Markov Chains

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# Overview

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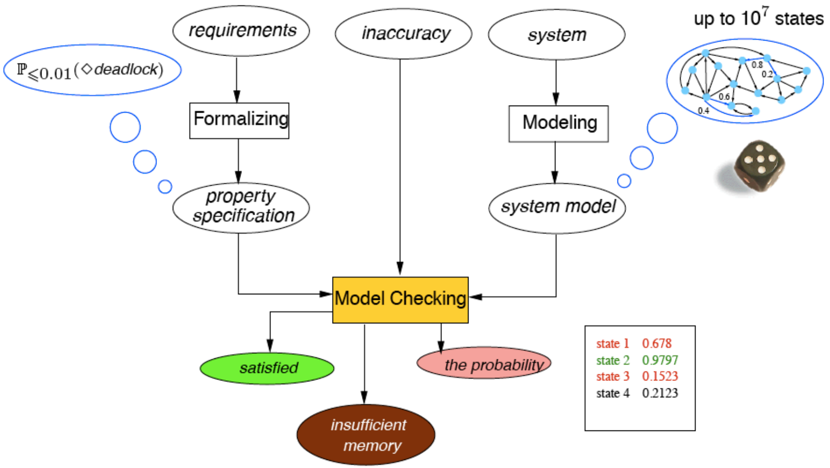
  - randomized algorithms
- When problems are undecidable deterministically
  - repeated reachability of lossy channel systems, ...

# What is probabilistic model checking?



# Probabilistic models

| | Nondeterminism no | Nondeterminism yes |
|---|---|---|
| Discrete time | discrete-time Markov chain (DTMC) | Markov decision process (MDP) |
| Continuous time | CTMC | CTMDP |

Some other models: probabilistic variants of (priced) timed automata

# Probability theory is simple, isn't it?

*In no other branch of mathematics*
*is it so easy to make mistakes*
*as in probability theory*

Henk Tijms, "Understanding Probability" (2004)

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# DTMCs — A transition system perspective

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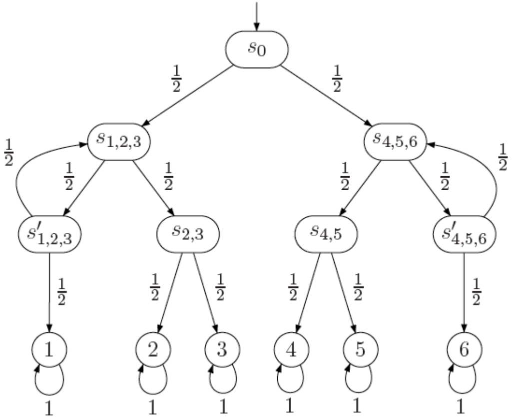
## Initial states

- $\iota_{\text{init}}(s)$ is the probability that DTMC $\mathcal{D}$ starts in state $s$
- the set $\{ s \in S \mid \iota_{\text{init}}(s) > 0 \}$ are the possible initial states.

# Simulating a die by a fair coin [Knuth & Yao]

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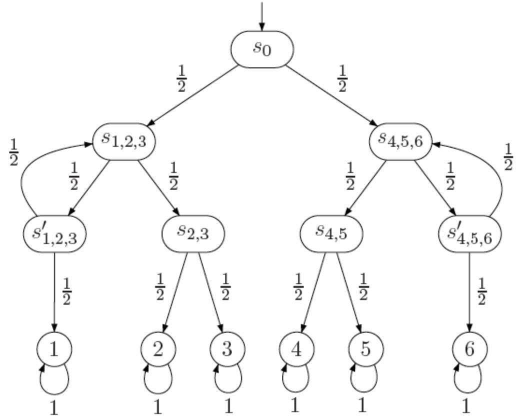


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A geometric distribution is the only discrete probability distribution that is memoryless.

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The probability to move from $s$ to $s'$ in $n \in \mathbb{N}$ steps is inductively defined:

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Repeating this scheme: $\mathbf{P}^{(n)} = \mathbf{P} \cdot \mathbf{P}^{(n-1)} = \ldots = \mathbf{P}^{n-1} \cdot \mathbf{P}^{(1)} = \mathbf{P}^{n}$.

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When considering $\Theta_n^{\mathcal{D}}$ as vector $(\Theta_n^{\mathcal{D}})_{t \in S}$ we have:

$$\Theta_n^{\mathcal{D}} \;=\; \iota_{\mathrm{init}} \cdot \underbrace{\mathbf{P} \cdot \mathbf{P} \cdot \ldots \cdot \mathbf{P}}_{n \text{ times}} \;=\; \iota_{\mathrm{init}} \cdot \mathbf{P}^n.$$

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3. . . . and they are transient probabilities in a slightly modified DTMC.

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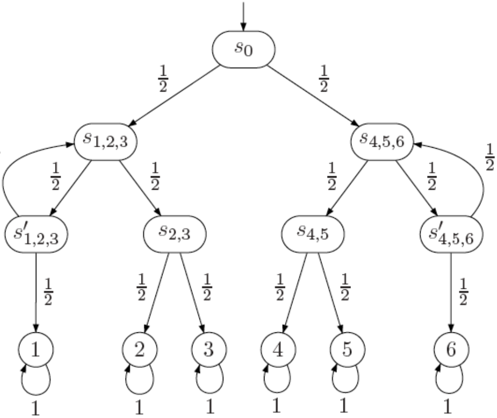


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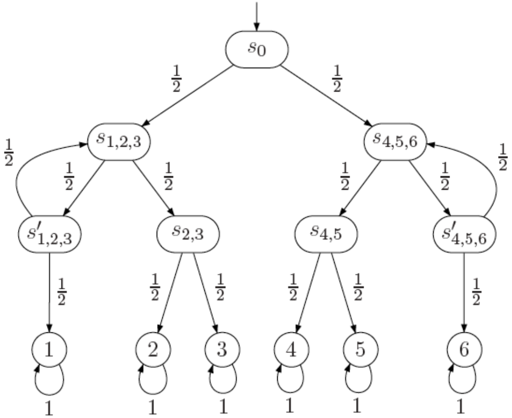


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$Paths(\mathcal{D})$ denotes the set of paths in $\mathcal{D}$, and $Paths^*(\mathcal{D})$ its finite prefixes.

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Repeatedly visit a state in $G$; formally:

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## Persistence

Eventually reach in a state in $G$ and always stay there; formally:

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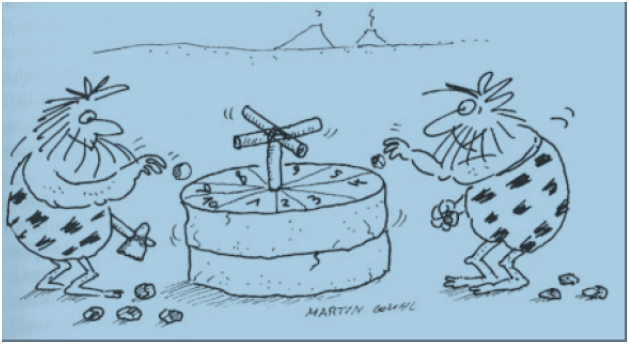
# What's the probability of infinite paths?



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- Cylinder set of a finite path := set of all its infinite continuations.

# Probability measure on DTMCs

## Cylinder set

The *cylinder set* of finite path $\hat{\pi} = s_0\, s_1 \ldots s_n \in Paths^*(\mathcal{D})$ is defined by:

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## Probability space of a DTMC

The set of events of the probability space DTMC $\mathcal{D}$ contains all cylinder sets $Cyl(\hat{\pi})$ where $\hat{\pi}$ ranges over all finite paths in $\mathcal{D}$.

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Events $\Diamond G$, $\Box G$, $\overline{F} \,\mathsf{U}\, G$, $\Box\Diamond G$ and $\Diamond\Box G$ are measurable on any DTMC.

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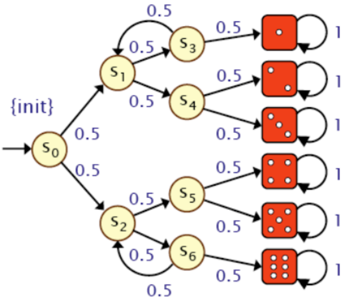
A similar proof strategy applies to the case $\overline{F} \, \mathsf{U} \, G$.

# Reachability probabilities: Knuth's die

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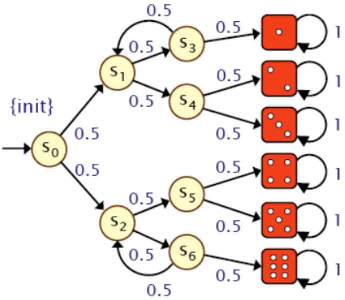


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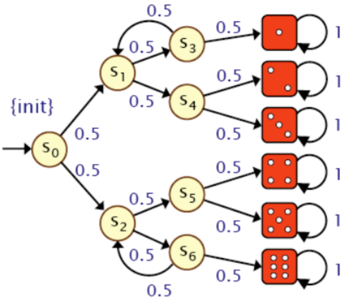


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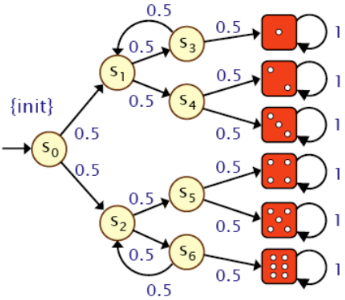


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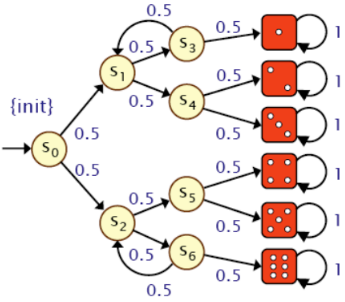


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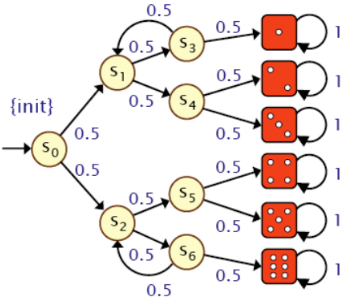


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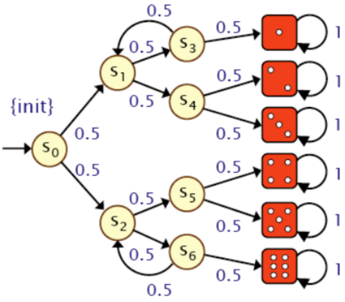


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There is however an simpler way to obtain reachability probabilities!

# Reachability probabilities in finite DTMCs

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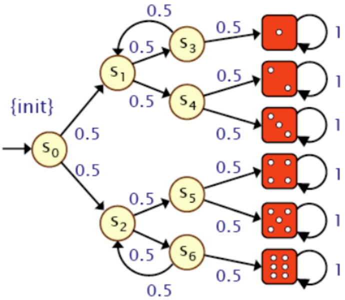
$$x_s = \underbrace{\sum_{t \in S \setminus G} \mathbf{P}(s,t) \cdot x_t}_{\text{reach } G \text{ via } t \in S \setminus G} + \underbrace{\sum_{u \in G} \mathbf{P}(s,u)}_{\text{reach } G \text{ in one step}}$$

# Reachability probabilities: Knuth's die

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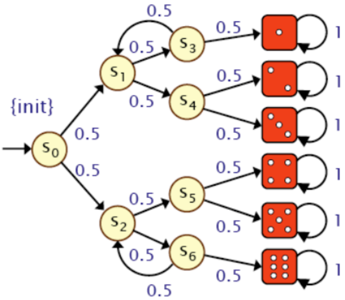


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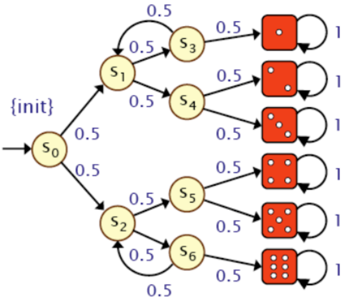


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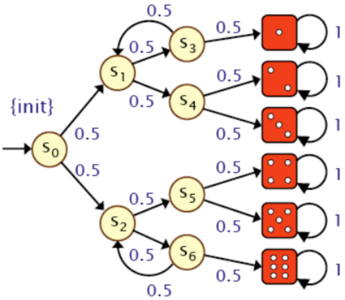


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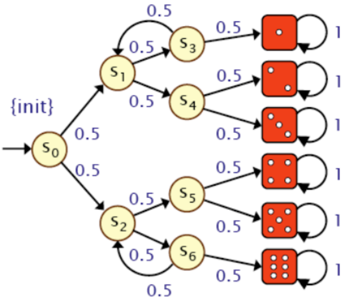


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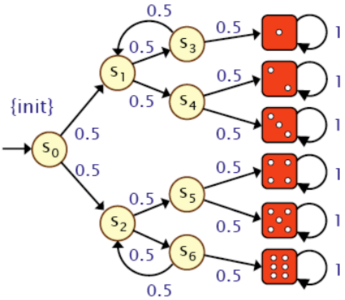


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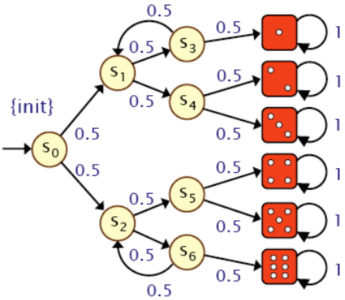


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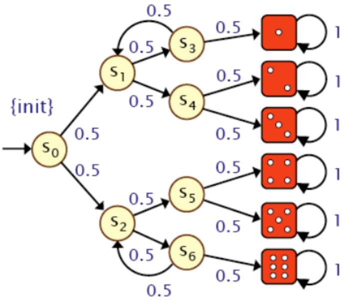


- Consider the event $\Diamond 4$
- Using the previous characterisation we obtain:

$$x_1 = x_2 = x_3 = x_5 = x_6 = 0 \text{ and } x_4 = 1$$

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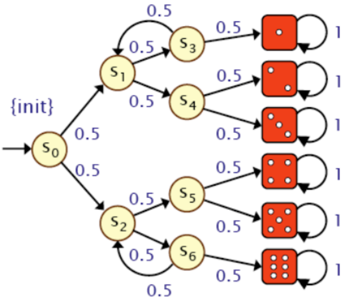
where $\mathbf{I}$ is the identity matrix of cardinality $|S_?| \times |S_?|$.

# Reachability probabilities: Knuth's die

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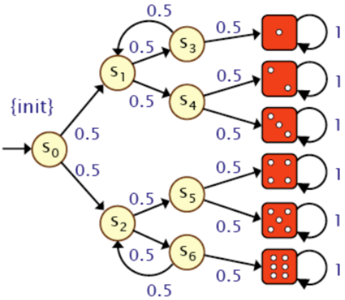


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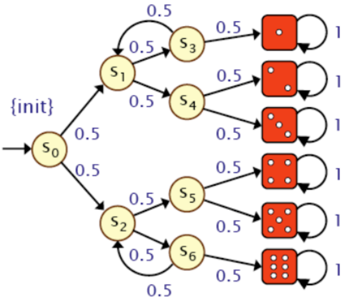


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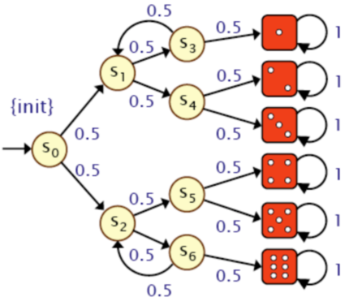


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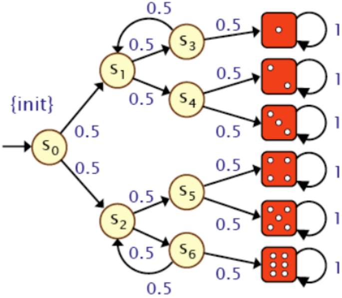


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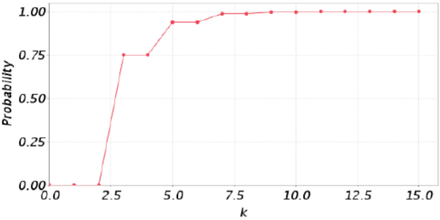
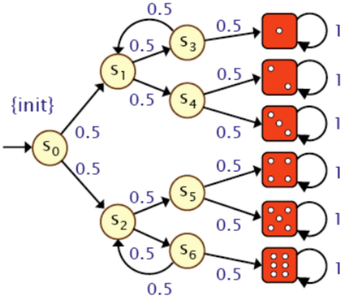
Alternatives: e.g., Jacobi or Gauss-Seidel, successive overrelaxation (SOR).

# Example: Knuth's die

- Let $G = \{1, 2, 3, 4, 5, 6\}$
- Then $Pr(s_0 \models \Diamond G) = 1$
- And $Pr(s_0 \models \Diamond^{\leqslant k} G)$
  for $k \in \mathbb{N}$ is given by:



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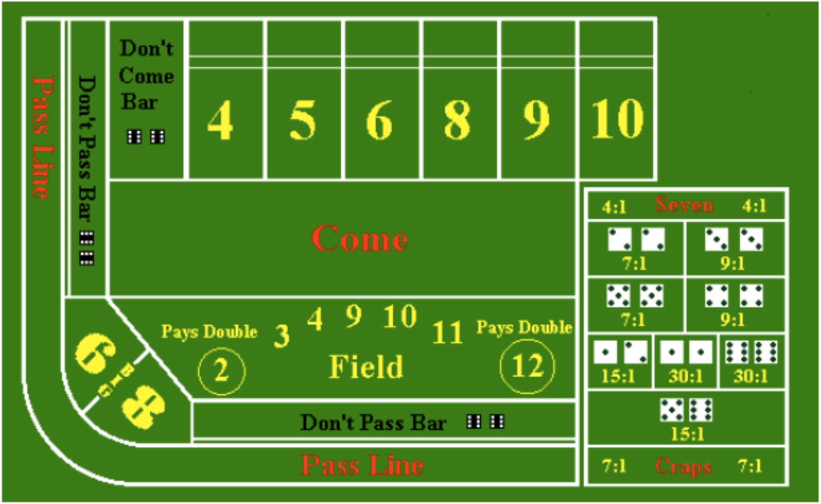
# Spare time tonight? Play Craps!



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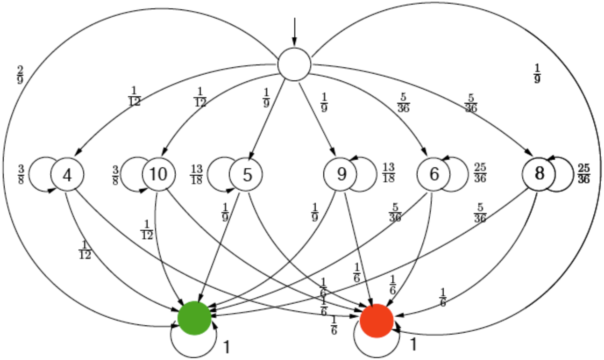


What is the probability to win the Craps game?

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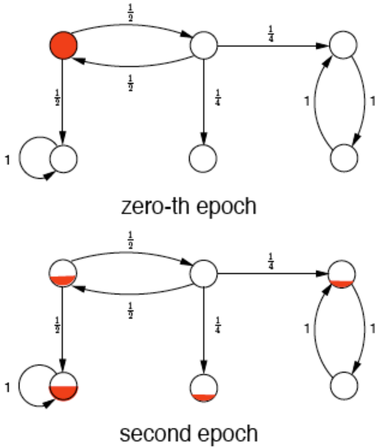
4. . . . and they are transient probabilities in a slightly modified DTMC.

# Where do we end up in the end?



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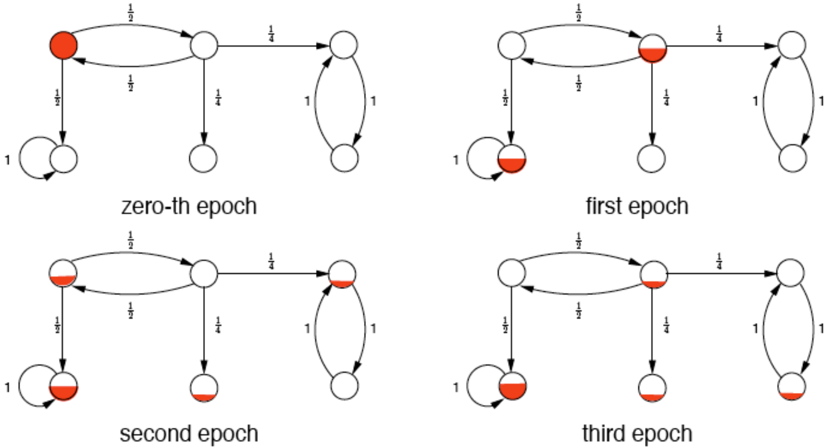
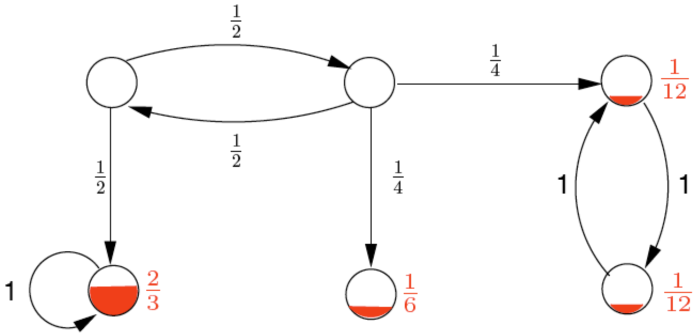



Which states have a probability $> 0$ when repeating this on the long run?

# On the long run



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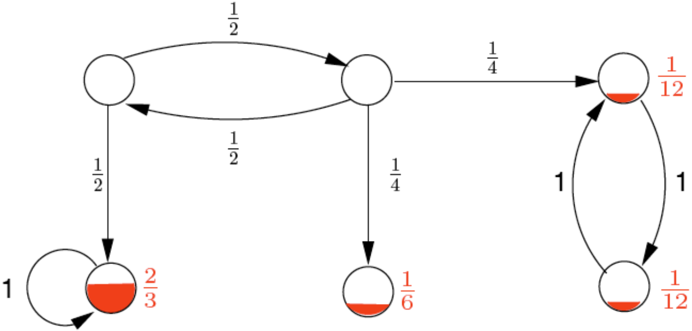


The probability mass on the long run is only left in bottom SCCs.

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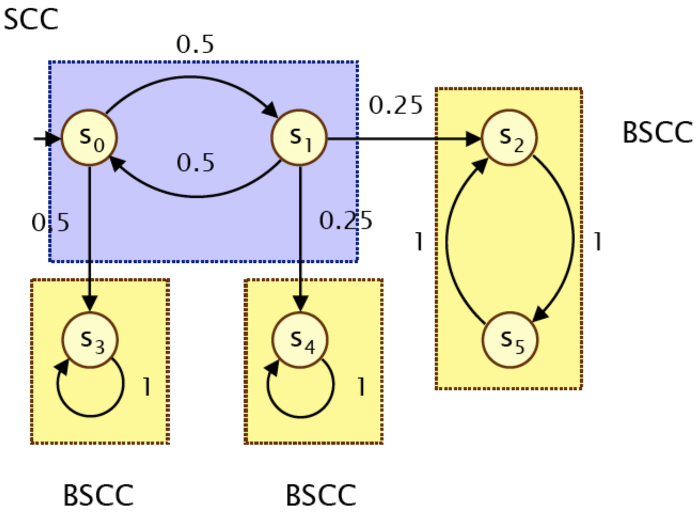
- SCC $T$ is a *bottom SCC* (BSCC) if no state outside $T$ is reachable from $T$, i.e., for any state $s \in T$, $\mathbf{P}(s, T) = \sum_{t \in T} \mathbf{P}(s, t) = 1$.

# Example



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## Remark

For any state $s$ in (possibly infinite) DTMC $\mathcal{D}$:

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