Modeling Concurrent and Probabilistic Systems

Lecture 7: Strong Simulation and Weak Bisimulation

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Winter Semester 2007/08

Outline

1. Strong Simulation
2. Definition of Weak Bisimulation
3. Properties of Weak Bisimulation

Strong Simulation

Observation: sometimes, the concept of strong bisimulation is too strong (example: extending a system by new features)

Definition 7.1 (Strong simulation)

A relation $\rho \subseteq Prc \times Prc$ is called a strong simulation if, whenever $P \rho Q$ and $P \xrightarrow{\alpha} P'$, there exists $Q' \in Prc$ such that $Q \xrightarrow{\alpha} Q'$ and $P' \rho Q'$.
We say that Q strongly simulates P if there exists a strong simulation ρ such that $P \rho Q$.

Thus: if Q strongly simulates P, then whatever transition path P takes, Q can match it by a path which retains all of P's options.

Example 7.2

Q strongly simulates P, but not vice versa

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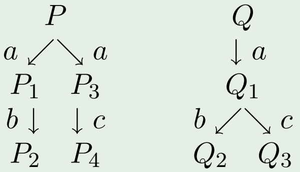

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Strong Simulation and Bisimulation

Corollary 7.3

If $P \sim Q$, then Q strongly simulates P, and P strongly simulates Q.

Proof.

A strong bisimulation $\rho \subseteq Prc \times Prc$ for $P \sim Q$ is a strong simulation for both directions. □

Caveat: the converse does generally not hold!

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Q simulates P and vice versa, but $P \not\sim Q$

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If $P \sim Q$, then Q strongly simulates P, and P strongly simulates Q.

Proof.

A strong bisimulation $\rho \subseteq \mathit{Prc} \times \mathit{Prc}$ for $P \sim Q$ is a strong simulation for both directions. □

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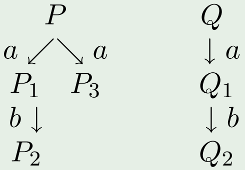

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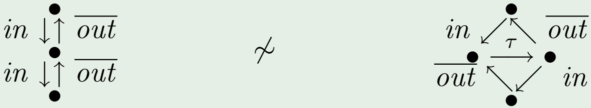
Outline

Example 7.5

Sequential and parallel two-place buffer:

$$
\begin{aligned}
B_0(in, out) &= in.B_1(in, out) \\
B_1(in, out) &= \overline{out}.B_0(in, out) + \\
&\quad\; in.B_2(in, out) \\
B_2(in, out) &= \overline{out}.B_1(in, out)
\end{aligned}
$$

$$
\begin{aligned}
B_{\|}(in, out) &= \mathsf{new}\ com\ (B(in, com) \parallel \\
&\qquad\qquad B(com, out)) \\
B(in, out) &= in.\overline{out}.B(in, out)
\end{aligned}
$$



Definition of Weak Bisimulation I

Idea: abstract from silent actions

Definition 7.6

- Given $w \in Act^*$, $\hat{w} \in (N \cup \overline{N})^*$ denotes the sequence of non-τ-actions in w (in particular, $\hat{\tau^n} = \varepsilon$ for every $n \in \mathbb{N}$).
- For $w = \alpha_1 \dots \alpha_n \in Act^*$ and $P, Q \in Prc$, we let

$$P \stackrel{w}{\Longrightarrow} Q \iff P \left(\stackrel{\tau}{\longrightarrow}\right)^* \stackrel{\alpha_1}{\longrightarrow} \left(\stackrel{\tau}{\longrightarrow}\right)^* \dots \left(\stackrel{\tau}{\longrightarrow}\right)^* \stackrel{\alpha_n}{\longrightarrow} \left(\stackrel{\tau}{\longrightarrow}\right)^* Q$$

  (and hence: $\stackrel{\varepsilon}{\Longrightarrow} = \left(\stackrel{\tau}{\longrightarrow}\right)^*$).
- A relation $\rho \subseteq Prc \times Prc$ is called a weak bisimulation if $P \rho Q$ implies, for every $\alpha \in Act$,
  1. $P \stackrel{\alpha}{\longrightarrow} P' \implies$ ex. $Q' \in Prc$ such that $Q \stackrel{\hat{\alpha}}{\Longrightarrow} Q'$ and $P' \rho Q'$
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- $P, Q \in Prc$ are called weakly bisimilar (notation: $P \approx Q$) if there exists a weak bisimulation ρ such that $P \rho Q$.

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Remark: each of the two clauses in the definition of weak bisimulation subsumes two cases:

- $P \xrightarrow{\alpha} P'$ where $\alpha \neq \tau$
  $\implies$ ex. $Q' \in Prc$ such that $Q\ (\xrightarrow{\tau})^* \xrightarrow{\alpha} (\xrightarrow{\tau})^*\ Q'$ and $P' \rho Q'$
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  (where $Q' = Q$ is admissible)

Example 7.7

on the board

Definition of Weak Bisimulation II

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Outline

Properties of Weak Bisimulation I

Theorem 7.8

$\approx$ is an equivalence relation.

Proof.

in analogy to the corresponding proof for $\sim$ (Theorem 4.2)

In particular, the following characterization is still valid:

$$\approx = \bigcup \{\rho \mid \rho \text{ weak bisimulation}\},$$

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Properties of Weak Bisimulation II

Moreover Definition 7.6 implies that every strong bisimulation is also a weak one (since, for every $\alpha \in Act$, $\xrightarrow{\alpha} \subseteq \xRightarrow{\hat{\alpha}}$). This yields the desired connection to LTS equivalence: for every $P, Q \in Prc$,

$$LTS(P) = LTS(Q) \implies P \sim Q \implies P \approx Q.$$

Furthermore trace equivalence is implied if the definition is adapted:

$$P \approx Q \implies \hat{Tr}(P) = \hat{Tr}(Q)$$

where $\hat{Tr}(P) := \{\hat{w} \mid w \in Tr(P)\} \subseteq (N \cup \overline{N})^*$.

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Properties of Weak Bisimulation III

Another important property is

Lemma 7.9

For every $P \in Prc$,

$$P \approx \tau.P$$

Proof.

We show that

$$\rho := \{(P, \tau.P)\} \cup id_{Prc}$$

is a weak bisimulation:

1. let $P \xrightarrow{\alpha} P'$
   $\implies \tau.P \xrightarrow{\tau} P \xrightarrow{\alpha} P'$
   $\implies \tau.P \stackrel{\hat{\alpha}}{\Longrightarrow} P'$ with $P' \rho P'$ (since $id_{Prc} \subseteq \rho$)
2. the only transition of $\tau.P$ is $\tau.P \xrightarrow{\tau} P$;
   it is simulated by $P \stackrel{\varepsilon}{\Longrightarrow} P$ with $P \rho P$

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Properties of Weak Bisimulation IV

Using Lemma 7.9, however, we can show that $\approx$ is not a congruence:

It is true that	$b.\mathsf{nil} \approx \tau.b.\mathsf{nil}$	(Theorem 7.8, Lemma 7.9)
but	$a.\mathsf{nil} + b.\mathsf{nil} \not\approx a.\mathsf{nil} + \tau.b.\mathsf{nil}$	(Example 7.7(b))

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Properties of Weak Bisimulation V

Also deadlock sensitivity is guaranteed if τ-actions are appropriately handled:

Theorem 7.10

Let $P, Q \in Prc$ such that $P \approx Q$. Then, for every $w \in (N \cup \overline{N})^*$,

$$P \stackrel{w}{\Longrightarrow} \not\longrightarrow \quad \iff \quad Q \stackrel{w}{\Longrightarrow} \not\longrightarrow .$$

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