Modeling Concurrent and Probabilistic Systems

Lecture 8: Observation Congruence

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Outline

1. Repetition: Weak Bisimulation
2. Further Properties of Weak Bisimulation
3. Definition of Observation Congruence
4. Properties of Observation Congruence

Repetition: Definition of Weak Bisimulation

Definition

- Given $w \in Act^*$, $\hat{w} \in (N \cup \overline{N})^*$ denotes the sequence of non-τ-actions in w (in particular, $\hat{\tau^n} = \varepsilon$ for every $n \in \mathbb{N}$).
- For $w = \alpha_1 \ldots \alpha_n \in Act^*$ and $P, Q \in Prc$, we let

$$P \stackrel{w}{\Longrightarrow} Q \iff P \left(\stackrel{\tau}{\longrightarrow}\right)^* \stackrel{\alpha_1}{\longrightarrow} \left(\stackrel{\tau}{\longrightarrow}\right)^* \ldots \left(\stackrel{\tau}{\longrightarrow}\right)^* \stackrel{\alpha_n}{\longrightarrow} \left(\stackrel{\tau}{\longrightarrow}\right)^* Q$$

  (and hence: $\stackrel{\varepsilon}{\Longrightarrow} = \left(\stackrel{\tau}{\longrightarrow}\right)^*$).
- A relation $\rho \subseteq Prc \times Prc$ is called a weak bisimulation if $P \rho Q$ implies, for every $\alpha \in Act$,
  1. $P \stackrel{\alpha}{\longrightarrow} P' \implies$ ex. $Q' \in Prc$ such that $Q \stackrel{\hat{\alpha}}{\Longrightarrow} Q'$ and $P' \rho Q'$
  2. $Q \stackrel{\alpha}{\longrightarrow} Q' \implies$ ex. $P' \in Prc$ such that $P \stackrel{\hat{\alpha}}{\Longrightarrow} P'$ and $P' \rho Q'$
- $P, Q \in Prc$ are called weakly bisimilar (notation: $P \approx Q$) if there exists a weak bisimulation ρ such that $P \rho Q$.

Repetition: Properties of Weak Bisimulation

Properties

1. $P \sim Q \implies P \approx Q$
2. $\approx$ is an equivalence relation
3. $LTS(P) = LTS(Q) \implies P \approx Q$
4. $P \approx Q \implies \hat{Tr}(P) = \hat{Tr}(Q)$
5. $\approx$ is (non-τ) deadlock sensitive
6. For every $P \in Prc$, $P \approx \tau.P$
7. $\approx$ is not a congruence:

   It is true that $b.\mathsf{nil} \approx \tau.b.\mathsf{nil}$
   but $a.\mathsf{nil} + b.\mathsf{nil} \not\approx a.\mathsf{nil} + \tau.b.\mathsf{nil}$

Outline

Further Properties of Weak Bisimulation

Lemma 8.1

For every $P, Q, R \in Prc$,

1. $P + Q \approx Q + P$
2. $P + (Q + R) \approx (P + Q) + R$
3. $P + \mathsf{nil} \approx P$
4. $P \parallel Q \approx Q \parallel P$
5. $P \parallel (Q \parallel R) \approx (P \parallel Q) \parallel R$
6. $P \parallel \mathsf{nil} \approx P$

Proof.

similar to Lemma 5.2 (strong bisimulation; omitted) □

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Proof.

similar to Lemma 5.2 (strong bisimulation; omitted) □

Outline

Definition of Observation Congruence I

Goal: introduce an equivalence which has most of the desirable properties of $\approx$ and which is preserved under all CCS operators

Definition 8.2

$P, Q \in Prc$ are called observationally congruent (notation: $P \simeq Q$) if, for every $\alpha \in Act$,

1. $P \xrightarrow{\alpha} P' \implies$ ex. $Q' \in Prc$ such that $Q \overset{\alpha}{\Longrightarrow} Q'$ and $P' \approx Q'$
2. $Q \xrightarrow{\alpha} Q' \implies$ ex. $P' \in Prc$ such that $P \overset{\alpha}{\Longrightarrow} P'$ and $P' \approx Q'$

Remark: $\simeq$ differs from $\approx$ only in the use of $\overset{\alpha}{\Longrightarrow}$ rather than $\overset{\hat{\alpha}}{\Longrightarrow}$, i.e., it requires τ-actions from P or Q to be simulated by at least one τ-step in the other process. This only applies to the first step; the successors just have to satisfy $P' \approx Q'$ (and not $P' \simeq Q'$).

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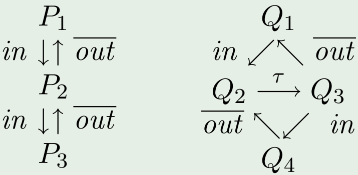
Example 8.3

1. Sequential and parallel two-place buffer:



$P_1 \simeq Q_1$ since $P_1 \approx Q_1$ (cf. Example 7.7) and neither P_1 nor Q_1 has initial τ-steps

2. $\tau.a.\mathsf{nil} \not\simeq a.\mathsf{nil}$
(since $\tau.a.\mathsf{nil} \xrightarrow{\tau}$ but $a.\mathsf{nil} \not\xrightarrow{\tau}$)

3. $a.\tau.\mathsf{nil} \simeq a.\mathsf{nil}$
(since $\tau.\mathsf{nil} \approx \mathsf{nil}$)

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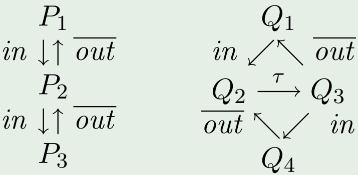

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   (since $\tau.\mathsf{nil} \approx \mathsf{nil}$)

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1. Sequential and parallel two-place buffer:

$$
\begin{array}{ccc}
 & P_1 & \\
in & \downarrow\uparrow & \overline{out} \\
 & P_2 & \\
in & \downarrow\uparrow & \overline{out} \\
 & P_3 &
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & Q_1 & & \\
 & in \ \swarrow & & \nwarrow \ \overline{out} & \\
Q_2 & & \xrightarrow{\tau} & & Q_3 \\
 & \overline{out} \ \nwarrow & & \swarrow \ in & \\
 & & Q_4 & &
\end{array}
$$

   $P_1 \simeq Q_1$ since $P_1 \approx Q_1$ (cf. Example 7.7) and neither P_1 nor Q_1 has initial τ-steps

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3. $a.\tau.\mathsf{nil} \simeq a.\mathsf{nil}$
   (since $\tau.\mathsf{nil} \approx \mathsf{nil}$)

Outline

Properties of Observation Congruence I

Corollary 8.4

For every $P, Q \in Prc$,

1. $P \sim Q \implies P \simeq Q$
2. $P \simeq Q \implies P \approx Q$

Proof.

1. since $\xrightarrow{\alpha} \subseteq \xRightarrow{\alpha}$ and $\sim \subseteq \approx$
2. since $\xRightarrow{\alpha} \subseteq \xRightarrow{\hat{\alpha}}$

□

Remark: this implies that

- processes with identical LTSs are $\simeq$-equivalent,
- $\simeq$-equivalent processes are (non-τ) trace equivalent, and
- $\simeq$ is (non-τ) deadlock sensitive.

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1. since $\xrightarrow{\alpha} \subseteq \overset{\alpha}{\Longrightarrow}$ and $\sim \subseteq \approx$
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- $\simeq$-equivalent processes are (non-τ) trace equivalent, and
- $\simeq$ is (non-τ) deadlock sensitive.

Properties of Observation Congruence II

Theorem 8.5

For every $P, Q \in Prc$,

$$P \simeq Q \iff P + R \approx Q + R \text{ for every } R \in Prc.$$

Proof.

on the board □

Remark: $\simeq$ is therefore the largest congruence contained in $\approx$

Properties of Observation Congruence II

Theorem 8.5

For every $P, Q \in \mathit{Prc}$,

$$P \simeq Q \iff P + R \approx Q + R \text{ for every } R \in \mathit{Prc}.$$

Proof.

on the board □

Remark: $\simeq$ is therefore the largest congruence contained in $\approx$

Properties of Observation Congruence II

Theorem 8.5

For every $P, Q \in Prc$,

$$P \simeq Q \iff P + R \approx Q + R \text{ for every } R \in Prc.$$

Proof.

on the board □

Remark: $\simeq$ is therefore the largest congruence contained in $\approx$

Properties of Observation Congruence III

Theorem 8.6

$\simeq$ *is an equivalence relation.*

Proof.

on the board □

Theorem 8.6

$\simeq$ *is an equivalence relation.*

Proof.

on the board □

Properties of Observation Congruence IV

Theorem 8.7

$\simeq$ *is a CCS congruence.*

Proof.

omitted □

Properties of Observation Congruence IV

Theorem 8.7

$\simeq$ *is a CCS congruence.*

Proof.

omitted $\square$

Theorem 8.8

For every $P, Q \in Prc$,

$$P \approx Q \iff P \simeq Q \text{ or } P \simeq \tau.Q \text{ or } \tau.P \simeq Q.$$

Proof.

see Exercise 5.3 □

Theorem 8.8

For every $P, Q \in Prc$,

$$P \approx Q \iff P \simeq Q \text{ or } P \simeq \tau.Q \text{ or } \tau.P \simeq Q.$$

Proof.

see Exercise 5.3 □