Software Modeling and Verification
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Modeling Concurrent and Probabilistic Systems

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– Solution 6 –

Exercise 1 (4 points)

a) $P \not\approx R$:
Assume there exists a weak bisimulation ρ with $P\rho R$.
Then $R \xrightarrow{\tau} R_2$ and hence there is P' such that $P \xRightarrow{\hat{\tau}} P'$.
Again, the only possible choice for P' is P (i.e. zero τ-steps).
By definition $P\rho R$ implies $P'\rho R_2$. However, $P' \xrightarrow{a} \mathsf{nil}$ whereas $R_2 \not\xRightarrow{\hat{a}}$.
$\Rightarrow$ Contradiction.

b) $Q \not\approx R$:
Assume there exists a weak bisimulation ρ with $Q\rho R$.
Then $R \xrightarrow{\tau} R_1$ and hence there exists Q' such that $Q \xRightarrow{\hat{\tau}} Q'$.
There are two choices for Q':

(a) Let $Q' = Q$ (i.e. zero τ-steps). Then $R_1 \rho Q$ is required by the definition of ρ.
Now $Q \xrightarrow{\tau} Q_1$. Hence there ex. R_1' such that $R_1 \xRightarrow{\hat{\tau}} R_1'$.
Only possible choice: $R_1' = R_1$. Now $R_1 \rho Q_1$. (*)
But $R_1 \xrightarrow{a}$ whereas $Q_1 \not\xRightarrow{\hat{a}}$ (the reverse holds for b).
$\Rightarrow$ Contradiction.

(b) Let $Q' = Q_1$. Then $R_1 \rho Q_1$ must be valid. $\Rightarrow$ Contradiction (see (*)).

Exercise 2 (6 points)

Let $P_1 \approx P_2$. Then there exists a weak bisimulation ρ with $P_1 \rho P_2$.

a) To show: $a.P_1 \approx a.P_2$ for all $a \in Act$.
Let $\rho' = \rho \cup \{(a.P_1, a.P_2) \mid a \in Act\}$.
It remains to show that ρ' is a weak bisimulation:
Obviously, ρ is a weak bisimulation. Further, for $a \in Act$,

- $a.P_1 \xrightarrow{a} P_1$ and $a.P_2 \xRightarrow{\hat{a}} P_2$ and $P_1 \rho' P_2$
- $a.P_2 \xrightarrow{a} P_2$ and $a.P_1 \xRightarrow{\hat{a}} P_1$ and $P_1 \rho' P_2$

$\Rightarrow \rho'$ is a weak bisimulation.

b) To show: $P_1 \mid\mid Q \approx P_2 \mid\mid Q$
$\rho' = \rho \cup \{(P_1 \mid\mid Q, P_2 \mid\mid Q) \mid P_1, P_2, Q \in Proc \text{ and } P_1 \approx P_2\}$
By definition, ρ is a weak bisimulation. Hence, for ρ', we have to show:

- $P_1 \mid\mid Q \xrightarrow{a} R_1' \quad \Rightarrow \exists R_2'$ such that $P_2 \mid\mid Q \xRightarrow{\hat{a}} R_2'$ and $R_1' \rho' R_2'$
- $P_2 \mid\mid Q \xrightarrow{a} R_2' \quad \Rightarrow \exists R_1'$ such that $P_1 \mid\mid Q \xRightarrow{\hat{a}} R_1$ and $R_1' \rho' R_2'$

Due to symmetry, we only consider one direction: Therefore distinguish three cases:

(a) $P_1 \xrightarrow{a} P_1'$. Then $P_1 \mid\mid Q \xrightarrow{a} P_1' \mid\mid Q$. As $P_1 \rho P_2$, there exists P_2' such that $P_2 \stackrel{\hat{a}}{\Longrightarrow} P_2'$ and $P_1' \rho P_2'$. Hence $P_2 \mid\mid Q \stackrel{\hat{a}}{\Longrightarrow} P_2' \mid\mid Q$. Then, by definition, $(P_1' \mid\mid Q)\rho'(P_2' \mid\mid Q)$.

(b) $Q \xrightarrow{a} Q'$. Then $P_1 \mid\mid Q \xrightarrow{a} P_1 \mid\mid Q'$ and also $P_2 \mid\mid Q \stackrel{\hat{a}}{\Longrightarrow} P_2 \mid\mid Q'$. As $P_1 \rho P_2$ and by definition of ρ', $(P_1 \mid\mid Q')\rho'(P_2 \mid\mid Q')$.

(c) $P_1 \mid\mid Q \xrightarrow{\tau} P_1' \mid\mid Q'$ where $P_1 \xrightarrow{a} P_1'$ and $Q \xrightarrow{\overline{a}} Q'$ for $a \in N \cup \overline{N}$.
Then, as $P_1 \rho P_2$ there ex. P_2' such that $P_2 \stackrel{\hat{a}}{\Longrightarrow} P_2'$ and $P_1' \rho P_2'$.
Therefore we have $P_2 \xrightarrow{\tau}^* R_1 \xrightarrow{a} R_2 \xrightarrow{\tau}^* P_2'$.
Thus $P_2 \mid\mid Q \xrightarrow{\tau}^* R_1 \mid\mid Q \xrightarrow{\tau} R_2 \mid\mid Q' \xrightarrow{\tau}^* P_2' \mid\mid Q'$.
Therefore $P_2 \mid\mid Q \stackrel{\hat{\tau}}{\Longrightarrow} P_2' \mid\mid Q'$. Further $(P_1' \mid\mid Q')\rho'(P_2' \mid\mid Q')$ for $P_1' \approx P_2'$.

c) To show: $\mathsf{new}\, a\, P_1 \approx \mathsf{new}\, a\, P_2$ for all $a \in N$.
Let $\rho' = \{(\mathsf{new}\, a\, P_1, \mathsf{new}\, a\, P_2) \mid a \in N, P_1 \approx P_2\}$.
Let $\mathsf{new}\, a\, P_1 \xrightarrow{\beta} \mathsf{new}\, a\, P_1'$ for $\beta \in Act$.
Then $P_1 \xrightarrow{\beta} P_1'$. Since $P_1 \rho P_2$ for some weak bisimulation ρ, there ex. P_2' such that $P_2 \stackrel{\hat{\beta}}{\Longrightarrow} P_2'$, i.e. $P_2 \xrightarrow{\tau}^* R_1 \xrightarrow{\beta} R_2 \xrightarrow{\tau}^* P_2'$ and $P_1' \rho P_2'$.
Thus $\mathsf{new}\, a\, P_2 \xrightarrow{\tau}^* \mathsf{new}\, a\, R_1 \xrightarrow{\beta} \mathsf{new}\, a\, R_2 \xrightarrow{\tau}^* \mathsf{new}\, a\, P_2'$.
Hence $\mathsf{new}\, a\, P_2 \stackrel{\hat{\beta}}{\Longrightarrow} \mathsf{new}\, a\, P_2'$ and $(\mathsf{new}\, a\, P_1', \mathsf{new}\, a\, P_2') \in \rho'$.

Exercise 3 (4 points)

a) $P \mid\mid \tau.Q \approx P \mid\mid Q$
Obviously, $Q \approx \tau.Q$. Then by 6.2b) it follows that $P \mid\mid Q \approx P \mid\mid \tau.Q$.

b) $P \mid\mid \tau.Q \not\simeq P \mid\mid Q$
Assume $a.\mathsf{nil} \mid\mid \tau.b.\mathsf{nil} \simeq a.\mathsf{nil} \mid\mid b.\mathsf{nil}$.
Then $a.\mathsf{nil} \mid\mid \tau.b.\mathsf{nil} \xrightarrow{\tau} a.\mathsf{nil} \mid\mid b.\mathsf{nil}$.
But $a.\mathsf{nil} \mid\mid b.\mathsf{nil} \stackrel{\tau}{\not\Longrightarrow}$ (since $a.\mathsf{nil} \mid\mid b.\mathsf{nil} \stackrel{\tau}{\Longrightarrow}$ means $a.\mathsf{nil} \mid\mid b.\mathsf{nil}(\xrightarrow{\tau}^*) \xrightarrow{\tau} (\xrightarrow{\tau}^*)$)
As $a.\mathsf{nil} \mid\mid b.\mathsf{nil}$ cannot execute a τ-step, $\not\simeq$.

c) $P \mid\mid \tau.Q \simeq \tau.(P \mid\mid Q)$ Three cases:

- $P \xrightarrow{a} P'$. Then $P \mid\mid \tau.Q \xrightarrow{a} P' \mid\mid \tau.Q$. Then $\tau.(P \mid\mid Q) \xrightarrow{\tau} P \mid\mid Q \xrightarrow{a} P' \mid\mid Q$.
  Therefore $\tau.(P \mid\mid Q) \stackrel{a}{\Longrightarrow} P' \mid\mid Q$.
  Obviously (by part (a)), $P' \mid\mid \tau.Q \approx P' \mid\mid Q$.
- $P \mid\mid \tau.Q \xrightarrow{\tau} P \mid\mid Q$. Then $\tau.(P \mid\mid Q) \stackrel{\tau}{\Longrightarrow} P \mid\mid Q$ and $P \mid\mid Q \approx P \mid\mid Q$.
- $\tau.(P \mid\mid Q) \xrightarrow{\tau} P \mid\mid Q$. But $P \mid\mid \tau.Q \stackrel{\tau}{\Longrightarrow} P \mid\mid Q$ (with $\tau.Q \xrightarrow{\tau} Q$) and $P \mid\mid Q \approx P \mid\mid Q$.