Software Modeling and Verification
Lehrstuhl für Informatik 2
RWTH Aachen University

Prof. Dr. Ir. J.-P. Katoen
Priv.-Doz. Dr. T. Noll
T. Han, D. Klink

Modeling Concurrent and Probabilistic Systems

Summer Term 09

– Series 8 –

Hand in until July 13 before the exercise class.

Exercise 1 (3 points)

Let $\mathcal{D} = (S, \mathbf{P})$ be an FPS and R an equivalence relation on S. R is a *weak bisimulation* on $\mathcal{D}$ if for all $s_1 R s_2$:

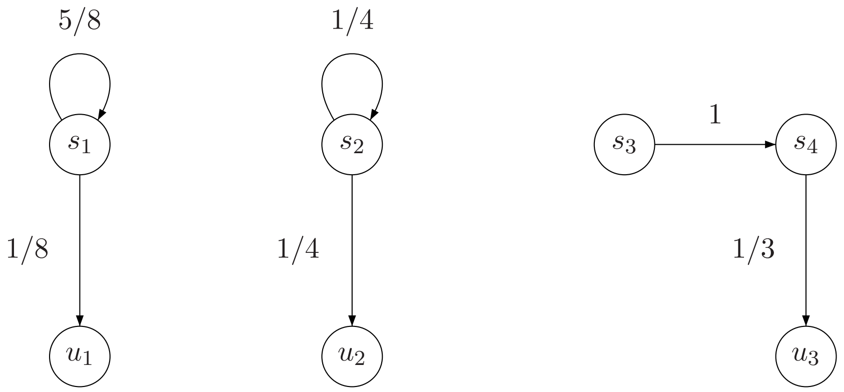
a) If $\mathbf{P}(s_i, [s_i]_R) < 1$ for $i = 1, 2$ then for all $C \in S/R$, $C \neq [s_1]_R = [s_2]_R$:

$$\frac{\mathbf{P}(s_1, C)}{1 - \mathbf{P}(s_1, [s_1]_R)} = \frac{\mathbf{P}(s_2, C)}{1 - \mathbf{P}(s_2, [s_2]_R)}.$$

b) s_1 can reach a state outside $[s_1]_R$ iff s_2 can reach a state outside $[s_2]_R$.

s_1 and s_2 in $\mathcal{D}$ are weakly bisimilar, denoted $s_1 \approx s_2$, iff there exists a weak bisimulation R on $\mathcal{D}$ such that $s_1 R s_2$.

Consider the FPS depicted in the figure. Find an equivalence relation $R \subseteq S \times S$ which is a weak bisimulation. Prove your answer! (Note: the depicted FPS is a single FPS!)



Solution The equivalence relation R with $S/R = \{\{s_1, s_2, s_3, s_4\}, \{u_1, u_2, u_3\}\}$ is a weak bisimulation. This can be seen as follows.

- For $i = 1, 2, 4$, $\mathbf{P}(s_i, [s_i]_R) < 1$, then for $C = \{u_1, u_2, u_3\}$, we have
$$\frac{\mathbf{P}(s_1, C)}{1 - \mathbf{P}(s_1, C)} = \frac{1/8}{1 - 5/8} = \frac{1/4}{1 - 1/4} = \frac{\mathbf{P}(s_2, C)}{1 - \mathbf{P}(s_2, C)} = \frac{1/3}{1} = \frac{\mathbf{P}(s_4, C)}{1 - \mathbf{P}(s_4, C)}.$$
- For s_3, since s_3 can reach a state outside $[s_3]$ as s_1, s_2 and s_4.
- For u_1, u_2, u_3, $\mathbf{P}(u_i, [u_i]_R) = 0$, then for $C' = \{s_1, s_2, s_3, s_4\}$, we have
$$\frac{\mathbf{P}(u_1, C')}{1 - \mathbf{P}(u_1, C')} = \frac{\mathbf{P}(u_2, C')}{1 - \mathbf{P}(u_2, C')} = \frac{\mathbf{P}(u_3, C')}{1 - \mathbf{P}(u_3, C')} = 0.$$

It follows that $s_1 \approx s_2 \approx s_3 \approx s_4$, $u_1 \approx u_2 \approx u_3$.

□

Exercise 2 **(4 points)**

Determine the semantics of X in terms of a probabilistic transition system:

$$X = \alpha.X \oplus_{\frac{1}{4}} (\beta.Y \oplus_{\frac{2}{3}} Y)$$

$$Y = \alpha.Y \oplus_{\frac{1}{3}} (\beta.X \oplus_{\frac{3}{4}} X)$$

Solution

- The approach of using indices:

$$\begin{array}{c}
\hline
\alpha.X \xrightarrow{(\alpha,1)}_{0} X \\
\hline
\alpha.X \oplus_{\frac{1}{4}} (\beta.Y \oplus_{\frac{2}{3}} Y) \xrightarrow{(\alpha,\frac{1}{4})}_{1.0} X \\
\hline
X \xrightarrow{(\alpha,\frac{1}{4})}_{1.0} X \\
\hline
Y \xrightarrow{(\alpha,\frac{1}{4}\cdot\frac{1}{6})}_{3.1.0} X \\
\hline
X \xrightarrow{(\alpha,\frac{1}{4}\cdot\frac{1}{6}\cdot\frac{1}{4})}_{3.3.1.0} X \\
\hline
Y \xrightarrow{(\alpha,\frac{1}{4}\cdot\frac{1}{6}\cdot\frac{1}{4}\cdot\frac{1}{6})}_{3.3.3.1.0} X \\
\hline
\vdots
\end{array}
\qquad
\begin{array}{c}
\hline
\alpha.Y \xrightarrow{(\alpha,1)}_{0} Y \\
\hline
\alpha.Y \oplus_{\frac{1}{3}} (\beta.X \oplus_{\frac{3}{4}} X) \xrightarrow{(\alpha,\frac{1}{3})}_{1.0} Y \\
\hline
Y \xrightarrow{(\alpha,\frac{1}{3})}_{1.0} Y \\
\hline
X \xrightarrow{(\alpha,\frac{1}{3}\cdot\frac{1}{4})}_{3.1.0} Y \\
\hline
Y \xrightarrow{(\alpha,\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{6})}_{3.3.1.0} Y \\
\hline
X \xrightarrow{(\alpha,\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{6}\cdot\frac{1}{4})}_{3.3.3.1.0} Y \\
\hline
\vdots
\end{array}$$

$$\begin{array}{c}
\hline
\beta.Y \xrightarrow{(\beta,1)}_{0} Y \\
\hline
\alpha.X \oplus_{\frac{1}{4}} (\beta.Y \oplus_{\frac{2}{3}} Y) \xrightarrow{(\beta,\frac{1}{2})}_{2.0} Y \\
\hline
X \xrightarrow{(\beta,\frac{1}{2})}_{2.0} Y \\
\hline
Y \xrightarrow{(\beta,\frac{1}{2}\cdot\frac{1}{6})}_{3.2.0} Y \\
\hline
X \xrightarrow{(\beta,\frac{1}{2}\cdot\frac{1}{6}\cdot\frac{1}{4})}_{3.3.2.0} Y \\
\hline
Y \xrightarrow{(\beta,\frac{1}{2}\cdot\frac{1}{6}\cdot\frac{1}{4}\cdot\frac{1}{6})}_{3.3.3.2.0} Y \\
\hline
\vdots
\end{array}
\qquad
\begin{array}{c}
\hline
\beta.X \xrightarrow{(\beta,1)}_{0} X \\
\hline
\alpha.Y \oplus_{\frac{1}{3}} (\beta.X \oplus_{\frac{3}{4}} X) \xrightarrow{(\beta,\frac{1}{2})}_{2.0} X \\
\hline
Y \xrightarrow{(\beta,\frac{1}{2})}_{2.0} X \\
\hline
X \xrightarrow{(\beta,\frac{1}{2}\cdot\frac{1}{4})}_{3.2.0} X \\
\hline
Y \xrightarrow{(\beta,\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{6})}_{3.3.2.0} X \\
\hline
X \xrightarrow{(\beta,\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{6}\cdot\frac{1}{4})}_{3.3.3.2.0} X \\
\hline
\vdots
\end{array}$$

The (reachable part of the) PTS for X is $(\{X, Y\}, \{\alpha, \beta\}, \mathbf{P}, \{(3.)^n 1.0, (3.)^n 2.0 \mid n \in \mathbb{N}\}, X)$ where

$$\mathbf{P}(X, \alpha, (3.)^n 1.0, X) = \begin{cases} \frac{1}{4} \cdot \left(\frac{1}{24}\right)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{P}(X, \alpha, (3.)^n 1.0, Y) = \begin{cases} \frac{1}{3} \cdot \frac{1}{4} \cdot \left(\frac{1}{24}\right)^{(n-1)/2} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{P}(Y, \alpha, (3.)^n 1.0, Y) = \begin{cases} \frac{1}{3} \cdot \left(\frac{1}{24}\right)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{P}(Y, \alpha, (3.)^n 1.0, X) = \begin{cases} \frac{1}{4} \cdot \frac{1}{6} \cdot \left(\frac{1}{24}\right)^{(n-1)/2} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{P}(X,\beta,(3.)^n 2.0,Y) = \begin{cases} \frac{1}{2}\cdot\left(\frac{1}{24}\right)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{P}(Y,\beta,(3.)^n 2.0,X) = \begin{cases} \frac{1}{2}\cdot\left(\frac{1}{24}\right)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{P}(X,\beta,(3.)^n 2.0,X) = \begin{cases} \frac{1}{2}\cdot\frac{1}{4}\cdot\left(\frac{1}{24}\right)^{(n-1)/2} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{P}(Y,\beta,(3.)^n 2.0,Y) = \begin{cases} \frac{1}{2}\cdot\frac{1}{6}\cdot\left(\frac{1}{24}\right)^{(n-1)/2} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

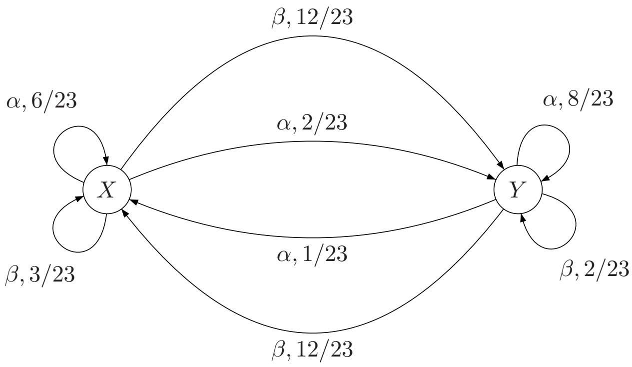
- The approach of solving a system of recursive equations yields:

$$\begin{aligned} \mathbf{P}(X,\alpha,X) &= 1/4 + 1/4\cdot\mathbf{P}(Y,\alpha,X) \\ \mathbf{P}(X,\alpha,Y) &= 1/4\cdot\mathbf{P}(Y,\alpha,Y) \\ \mathbf{P}(Y,\alpha,X) &= 1/6\cdot\mathbf{P}(X,\alpha,X) \\ \mathbf{P}(Y,\alpha,Y) &= 1/3 + 1/6\cdot\mathbf{P}(X,\alpha,Y) \\ \\ \mathbf{P}(X,\beta,X) &= 1/4\cdot\mathbf{P}(Y,\beta,X) \\ \mathbf{P}(X,\beta,Y) &= 1/2 + 1/4\cdot\mathbf{P}(Y,\beta,Y) \\ \mathbf{P}(Y,\beta,X) &= 1/2 + 1/6\cdot\mathbf{P}(X,\beta,X) \\ \mathbf{P}(Y,\beta,Y) &= 1/6\cdot\mathbf{P}(X,\beta,Y) \end{aligned}$$

with least solution

$$\begin{aligned} \mathbf{P}(X,\alpha,X) &= 6/23 \\ \mathbf{P}(X,\alpha,Y) &= 2/23 \\ \mathbf{P}(Y,\alpha,X) &= 1/23 \\ \mathbf{P}(Y,\alpha,Y) &= 8/23 \\ \\ \mathbf{P}(X,\beta,X) &= 3/23 \\ \mathbf{P}(X,\beta,Y) &= 12/23 \\ \mathbf{P}(Y,\beta,X) &= 12/23 \\ \mathbf{P}(Y,\beta,Y) &= 2/23 \end{aligned}$$

It is a PTS, so $\sum_{\gamma\in Act}\sum_{s'}\mathbf{P}(s,\gamma,s') = 1$, where $Act = \{\alpha,\beta\}$.



□

Exercise 3 **(3 points)**

For each of the following specifications, determine the semantics of X in terms of a probabilistic transition system with indices:

a) $X = \alpha.\big(\mathbf{new}\gamma\,([1/6]\alpha.\mathbf{0} + [1/3]\beta.0 + [1/2]\gamma.\mathbf{0})\big) \oplus_{3/4} \alpha.\mathbf{0}$

b) $X = \alpha.\mathbf{0} \oplus_{1/3} (\alpha.\alpha.\mathbf{0} \times X)$

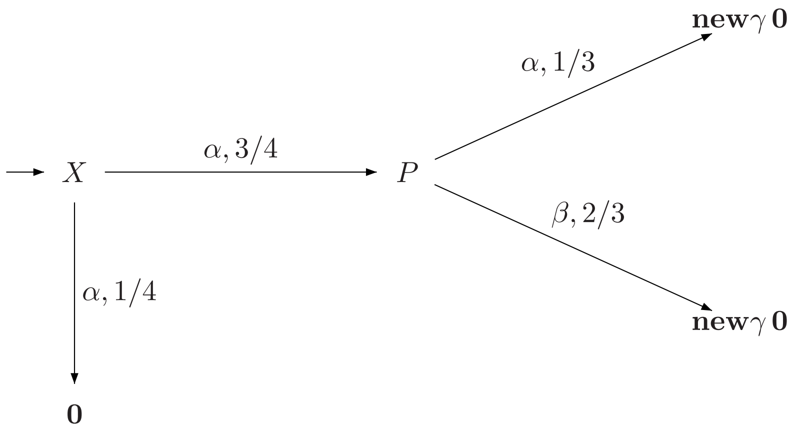
Solution

a) Set P to be the term $\mathbf{new}\gamma\,([1/6]\alpha.\mathbf{0} + [1/3]\beta.\mathbf{0} + [1/2]\gamma.\mathbf{0})$.

$$\frac{\dfrac{\dfrac{}{\alpha.P \xrightarrow{(\alpha,1)}_0 P}}{\alpha.P \oplus_{3/4} \alpha.\mathbf{0} \xrightarrow{(\alpha,3/4)}_{1.0} P}}{X \xrightarrow{(\alpha,3/4)}_{1.0} P}$$

$$\nu([\tfrac{1}{6}]\alpha.\mathbf{0} + [1/3]\beta.\mathbf{0} + [1/2]\gamma.\mathbf{0}, \{\gamma\}) = 1/2 \quad \frac{\dfrac{\dfrac{}{\alpha.\mathbf{0} \xrightarrow{(\alpha,1)}_0 \mathbf{0}}}{[1/6]\alpha.\mathbf{0} + [1/3]\beta.\mathbf{0} + [1/2]\gamma.\mathbf{0} \xrightarrow{(\alpha,1/6)}_{1.0} \mathbf{0}}}{P \xrightarrow{(\alpha,1/3)}_{1.0} \mathbf{new}\gamma\,\mathbf{0}}$$

$$\rightarrow X \xrightarrow{(\alpha,3/4)}_{1.0} P \xrightarrow{(\alpha,1/3)}_{1.0} \mathbf{new}\gamma\,\mathbf{0}$$



Note that $\mathbf{new}\gamma\,\mathbf{0} = \mathbf{0}$.

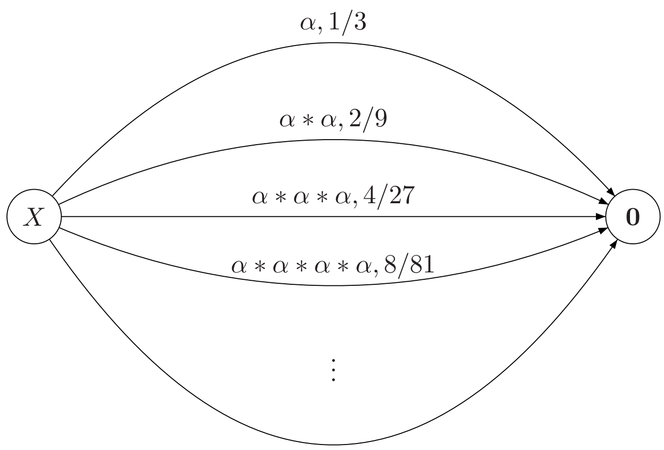
b) Let Y denote $\alpha.\alpha.\mathbf{0} \times X$. Note that $P \times \mathbf{0} = \mathbf{0} \times P = \mathbf{0}$, for any process P.

$$\dfrac{\dfrac{-}{\alpha.\mathbf{0} \xrightarrow{(\alpha,1)}_{0} \mathbf{0}}}{X \xrightarrow{(\alpha,1/3)}_{1.0} \mathbf{0}}$$

$$\dfrac{\dfrac{\alpha.\alpha.\mathbf{0} \xrightarrow{(\alpha,1)}_{0} \alpha.\mathbf{0} \qquad X \xrightarrow{(\alpha,1/3)}_{1.0} \mathbf{0}}{Y \xrightarrow{(\alpha*\alpha,1/3)}_{(0,1.0)} \alpha.\mathbf{0} \times \mathbf{0}}}{X \xrightarrow{(\alpha*\alpha,1/3\cdot 2/3)}_{2.(0,1.0)} \alpha.\mathbf{0} \times \mathbf{0}}$$

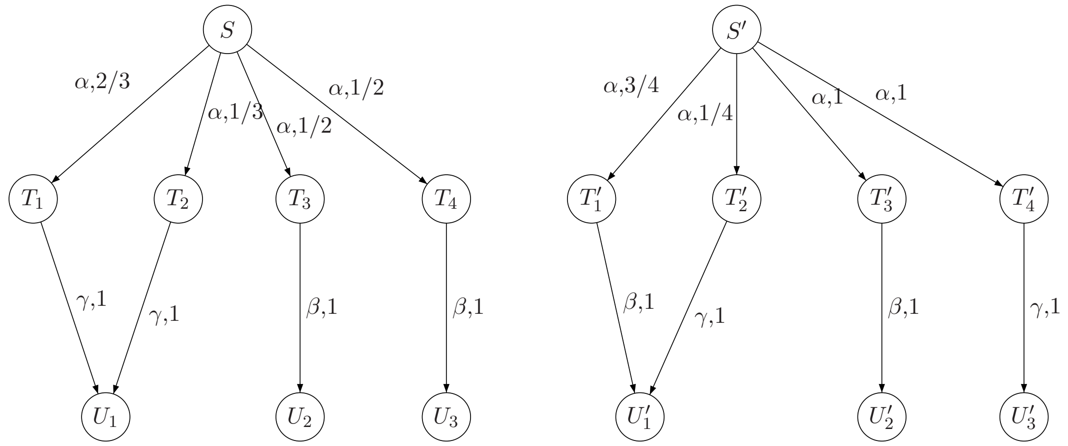
$$\dfrac{\dfrac{\alpha.\alpha.\mathbf{0} \xrightarrow{(\alpha,1)}_{0} \alpha.\mathbf{0} \qquad X \xrightarrow{(\alpha*\alpha,1/3\cdot 2/3)}_{2.(0,1.0)} \alpha.\mathbf{0} \times \mathbf{0}}{Y \xrightarrow{(\alpha*(\alpha*\alpha),1/3\cdot 2/3)}_{(0,2.(0,1.0))} \alpha.\mathbf{0} \times (\alpha.\mathbf{0} \times \mathbf{0})}}{X \xrightarrow{(\alpha*(\alpha*\alpha),1/3\cdot 2/3\cdot 2/3)}_{2.(0,2.(0,1.0))} \alpha.\mathbf{0} \times (\alpha.\mathbf{0} \times \mathbf{0})}$$

$$\vdots$$



Exercise 4 **(2 points)**

Consider two process automata PA_1, PA_2 given by the following figure:



Do we have

- $S \sim_p S'$?
- $S \simeq_p S'$?
- $S \sim_{cp} S'$?
- $S \simeq_{cp} S'$?

Solution Let $\mu_{1\alpha} = (2/3T_1, 1/3T_2)$, $\mu_{2\alpha} = (1/2T_3, 1/2T_4)$; $\mu_{3\alpha} = (3/4T_1{}', 1/4T_2{}')$, $\mu_{4\alpha} = (1T_3{}')$, $\mu_{5\alpha} = (1T_4{}')$ and let $C_1 = \{T_1, T_2, T_2', T_4'\}$ and $C_2 = \{T_3, T_4, T_1', T_3'\}$. It is easy to see that C_1 and C_2 are equivalent classes. The subscript α will be omitted when clear from context.

- No. Since $\mu_3(C_2) = 3/4$ and $\mu_1(C_2) = 1$, it follows that $\mu_3 \not\equiv \mu_1$. Therefore $S \not\sim_p S'$.
- No. Since $\mu_1 \not\sqsubseteq \mu_3$, therefore $S \not\simeq_p S'$.
- Yes.

  For $\mu_{3\alpha} = (3/4T_1{}', 1/4T_2{}')$, the combined transition is as follows: there exists $r_1 = 1/4, r_2 = 3/4$, such that

  $$\mu_{3\alpha} \sqsubseteq_{cp} r_1 \cdot \mu_{1\alpha} + r_2 \cdot \mu_{2\alpha}$$

  For the rest cases, similar. In total we have:

  $$\begin{aligned} \mu_1 &\equiv_{cp} \mu_5 \\ \mu_2 &\equiv_{cp} \mu_4 \\ \mu_3 &\equiv_{cp} 1/4 \cdot \mu_1 + 3/4 \cdot \mu_2 \end{aligned}$$

  Therefore $S \sim_{cp} S'$.
- Yes. Since $S \sim_{cp} S' \implies S \simeq_{cp} S'$, it directly holds that $S \simeq_{cp} S'$.