Modeling Concurrent and Probabilistic Systems

Lecture 8: Weak Bisimulation

Joost-Pieter Katoen   Thomas Noll

Software Modeling and Verification Group
RWTH Aachen University
noll@cs.rwth-aachen.de

http://www-i2.informatik.rwth-aachen.de/i2/mcps09/

Summer Semester 2009

Outline

Repetition: Definition of Strong Bisimulation

Definition (Strong bisimulation)

A relation $\rho \subseteq Prc \times Prc$ is called a strong bisimulation if $P \rho Q$ implies, for every $\alpha \in Act$,

1. $P \xrightarrow{\alpha} P' \implies$ ex. $Q' \in Prc$ such that $Q \xrightarrow{\alpha} Q'$ and $P' \rho Q'$
2. $Q \xrightarrow{\alpha} Q' \implies$ ex. $P' \in Prc$ such that $P \xrightarrow{\alpha} P'$ and $P' \rho Q'$

$P, Q \in Prc$ are called strongly bisimilar (notation: $P \sim Q$) if there exists a strong bisimulation ρ such that $P \rho Q$.

Repetition: Properties of Strong Bisimulation

Properties of strong bisimulation

1. $\sim$ is an equivalence relation
2. $LTS(P) = LTS(Q) \implies P \sim Q$
3. $P \sim Q \implies Tr(P) = Tr(Q)$
4. $\sim$ is a CCS congruence
5. $\sim$ is deadlock sensitive
6. $\sim$ is decidable for finite-state processes

Corollary

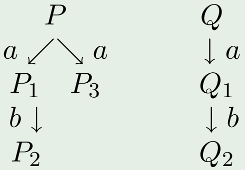
If $P \sim Q$, then Q strongly simulates P, and P strongly simulates Q.

Proof.

A strong bisimulation $\rho \subseteq Prc \times Prc$ for $P \sim Q$ is a strong simulation for both directions. □

Caveat: the converse does generally not hold!

Example



Q simulates P and vice versa, but $P \not\sim Q$

Outline

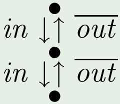
Inadequacy of Strong Bisimulation

Observation: requirement of exact matching sometimes too strong

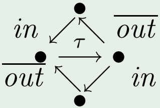
Example 8.1

Sequential and parallel two-place buffer:

$$
\begin{aligned}
B_0(in, out) &= in.B_1(in, out) \\
B_1(in, out) &= \overline{out}.B_0(in, out) + \\
&\quad\; in.B_2(in, out) \\
B_2(in, out) &= \overline{out}.B_1(in, out)
\end{aligned}
\qquad
\begin{aligned}
B_{\|}(in, out) &= \mathsf{new}\ com\ (B(in, com) \parallel \\
&\qquad\qquad B(com, out)) \\
B(in, out) &= in.\overline{out}.B(in, out)
\end{aligned}
$$



$\not\sim$



Definition of Weak Bisimulation I

Idea: abstract from silent actions

Definition 8.2

- Given $w \in Act^*$, $\widehat{w} \in (N \cup \overline{N})^*$ denotes the sequence of non-τ-actions in w (in particular, $\widehat{\tau^n} = \varepsilon$ for every $n \in \mathbb{N}$).
- For $w = \alpha_1 \ldots \alpha_n \in Act^*$ and $P, Q \in Prc$, we let

$$P \stackrel{w}{\Longrightarrow} Q \iff P \left(\stackrel{\tau}{\longrightarrow}\right)^* \stackrel{\alpha_1}{\longrightarrow} \left(\stackrel{\tau}{\longrightarrow}\right)^* \ldots \left(\stackrel{\tau}{\longrightarrow}\right)^* \stackrel{\alpha_n}{\longrightarrow} \left(\stackrel{\tau}{\longrightarrow}\right)^* Q$$

  (and hence: $\stackrel{\varepsilon}{\Longrightarrow} = \left(\stackrel{\tau}{\longrightarrow}\right)^*$).
- A relation $\rho \subseteq Prc \times Prc$ is called a weak bisimulation if $P \rho Q$ implies, for every $\alpha \in Act$,
  1. $P \stackrel{\alpha}{\longrightarrow} P' \implies$ ex. $Q' \in Prc$ such that $Q \stackrel{\widehat{\alpha}}{\Longrightarrow} Q'$ and $P' \rho Q'$
  2. $Q \stackrel{\alpha}{\longrightarrow} Q' \implies$ ex. $P' \in Prc$ such that $P \stackrel{\widehat{\alpha}}{\Longrightarrow} P'$ and $P' \rho Q'$
- $P, Q \in Prc$ are called weakly bisimilar (notation: $P \approx Q$) if there exists a weak bisimulation ρ such that $P \rho Q$.

Definition of Weak Bisimulation I

Idea: abstract from silent actions

Definition 8.2

- Given $w \in Act^*$, $\widehat{w} \in (N \cup \overline{N})^*$ denotes the sequence of non-τ-actions in w (in particular, $\widehat{\tau^n} = \varepsilon$ for every $n \in \mathbb{N}$).
- For $w = \alpha_1 \ldots \alpha_n \in Act^*$ and $P, Q \in Prc$, we let

$$P \overset{w}{\Longrightarrow} Q \iff P \left(\overset{\tau}{\longrightarrow}\right)^* \overset{\alpha_1}{\longrightarrow} \left(\overset{\tau}{\longrightarrow}\right)^* \ldots \left(\overset{\tau}{\longrightarrow}\right)^* \overset{\alpha_n}{\longrightarrow} \left(\overset{\tau}{\longrightarrow}\right)^* Q$$

(and hence: $\overset{\varepsilon}{\Longrightarrow} = \left(\overset{\tau}{\longrightarrow}\right)^*$).

- A relation $\rho \subseteq Prc \times Prc$ is called a weak bisimulation if $P \rho Q$ implies, for every $\alpha \in Act$,
  1. $P \overset{\alpha}{\longrightarrow} P' \implies$ ex. $Q' \in Prc$ such that $Q \overset{\widehat{\alpha}}{\Longrightarrow} Q'$ and $P' \rho Q'$
  2. $Q \overset{\alpha}{\longrightarrow} Q' \implies$ ex. $P' \in Prc$ such that $P \overset{\widehat{\alpha}}{\Longrightarrow} P'$ and $P' \rho Q'$
- $P, Q \in Prc$ are called weakly bisimilar (notation: $P \approx Q$) if there exists a weak bisimulation ρ such that $P \rho Q$.

Definition of Weak Bisimulation I

Idea: abstract from silent actions

Definition 8.2

- Given $w \in Act^*$, $\widehat{w} \in (N \cup \overline{N})^*$ denotes the sequence of non-τ-actions in w (in particular, $\widehat{\tau^n} = \varepsilon$ for every $n \in \mathbb{N}$).
- For $w = \alpha_1 \ldots \alpha_n \in Act^*$ and $P, Q \in Prc$, we let

$$P \overset{w}{\Longrightarrow} Q \iff P \left(\overset{\tau}{\longrightarrow}\right)^* \overset{\alpha_1}{\longrightarrow} \left(\overset{\tau}{\longrightarrow}\right)^* \ldots \left(\overset{\tau}{\longrightarrow}\right)^* \overset{\alpha_n}{\longrightarrow} \left(\overset{\tau}{\longrightarrow}\right)^* Q$$

  (and hence: $\overset{\varepsilon}{\Longrightarrow} = \left(\overset{\tau}{\longrightarrow}\right)^*$).
- A relation $\rho \subseteq Prc \times Prc$ is called a weak bisimulation if $P \rho Q$ implies, for every $\alpha \in Act$,
  1. $P \overset{\alpha}{\longrightarrow} P' \implies$ ex. $Q' \in Prc$ such that $Q \overset{\widehat{\alpha}}{\Longrightarrow} Q'$ and $P' \rho Q'$
  2. $Q \overset{\alpha}{\longrightarrow} Q' \implies$ ex. $P' \in Prc$ such that $P \overset{\widehat{\alpha}}{\Longrightarrow} P'$ and $P' \rho Q'$
- $P, Q \in Prc$ are called weakly bisimilar (notation: $P \approx Q$) if there exists a weak bisimulation ρ such that $P \rho Q$.

Definition of Weak Bisimulation I

Idea: abstract from silent actions

Definition 8.2

- Given $w \in Act^*$, $\widehat{w} \in (N \cup \overline{N})^*$ denotes the sequence of non-τ-actions in w (in particular, $\widehat{\tau^n} = \varepsilon$ for every $n \in \mathbb{N}$).
- For $w = \alpha_1 \ldots \alpha_n \in Act^*$ and $P, Q \in Prc$, we let

$$P \stackrel{w}{\Longrightarrow} Q \iff P \left(\stackrel{\tau}{\longrightarrow}\right)^* \stackrel{\alpha_1}{\longrightarrow} \left(\stackrel{\tau}{\longrightarrow}\right)^* \ldots \left(\stackrel{\tau}{\longrightarrow}\right)^* \stackrel{\alpha_n}{\longrightarrow} \left(\stackrel{\tau}{\longrightarrow}\right)^* Q$$

  (and hence: $\stackrel{\varepsilon}{\Longrightarrow} = \left(\stackrel{\tau}{\longrightarrow}\right)^*$).
- A relation $\rho \subseteq Prc \times Prc$ is called a weak bisimulation if $P \rho Q$ implies, for every $\alpha \in Act$,
  1. $P \stackrel{\alpha}{\longrightarrow} P' \implies$ ex. $Q' \in Prc$ such that $Q \stackrel{\widehat{\alpha}}{\Longrightarrow} Q'$ and $P' \rho Q'$
  2. $Q \stackrel{\alpha}{\longrightarrow} Q' \implies$ ex. $P' \in Prc$ such that $P \stackrel{\widehat{\alpha}}{\Longrightarrow} P'$ and $P' \rho Q'$
- $P, Q \in Prc$ are called weakly bisimilar (notation: $P \approx Q$) if there exists a weak bisimulation ρ such that $P \rho Q$.

Definition of Weak Bisimulation II

Remark: each of the two clauses in the definition of weak bisimulation subsumes two cases:

- $P \xrightarrow{\alpha} P'$ where $\alpha \neq \tau$
  $\Longrightarrow$ ex. $Q' \in Prc$ such that $Q \; (\xrightarrow{\tau})^* \xrightarrow{\alpha} (\xrightarrow{\tau})^* \; Q'$ and $P' \rho Q'$
- $P \xrightarrow{\tau} P'$
  $\Longrightarrow$ ex. $Q' \in Prc$ such that $Q \; (\xrightarrow{\tau})^* \; Q'$ and $P' \rho Q'$
  (where $Q' = Q$ is admissible)

Example 8.3

1. Sequential and parallel two-place buffer
2. A counterexample

(on the board)

Definition of Weak Bisimulation II

Remark: each of the two clauses in the definition of weak bisimulation subsumes two cases:

- $P \xrightarrow{\alpha} P'$ where $\alpha \neq \tau$
  $\Longrightarrow$ ex. $Q' \in Prc$ such that $Q\ (\xrightarrow{\tau})^* \xrightarrow{\alpha} (\xrightarrow{\tau})^*\ Q'$ and $P' \rho Q'$
- $P \xrightarrow{\tau} P'$
  $\Longrightarrow$ ex. $Q' \in Prc$ such that $Q\ (\xrightarrow{\tau})^*\ Q'$ and $P' \rho Q'$
  (where $Q' = Q$ is admissible)

Example 8.3

1. Sequential and parallel two-place buffer
2. A counterexample

(on the board)

Outline

Properties of Weak Bisimulation I

Theorem 8.4

$\approx$ is an equivalence relation.

Proof.

in analogy to the corresponding proof for $\sim$ (Theorem 5.2)

In particular, the following characterization is still valid:

$$\approx = \bigcup \{\rho \mid \rho \text{ weak bisimulation}\},$$

i.e., $\approx$ is again itself a weak bisimulation. □

Properties of Weak Bisimulation I

Theorem 8.4

$\approx$ *is an equivalence relation.*

Proof.

in analogy to the corresponding proof for $\sim$ (Theorem 5.2)

In particular, the following characterization is still valid:

$$\approx = \bigcup\{\rho \mid \rho \text{ weak bisimulation}\},$$

i.e., $\approx$ is again itself a weak bisimulation. □

Properties of Weak Bisimulation II

Moreover Definition 8.2 implies that every strong bisimulation is also a weak one (since, for every $\alpha \in Act$, $\xrightarrow{\alpha} \subseteq \xRightarrow{\widehat{\alpha}}$). This yields the desired connection to LTS equivalence: for every $P, Q \in Prc$,

$$LTS(P) = LTS(Q) \implies P \sim Q \implies P \approx Q.$$

Furthermore trace equivalence is implied if the definition is adapted:

$$P \approx Q \implies \hat{Tr}(P) = \hat{Tr}(Q)$$

where $\hat{Tr}(P) := \{\widehat{w} \mid w \in Tr(P)\} \subseteq (N \cup \overline{N})^*$.

Properties of Weak Bisimulation II

Moreover Definition 8.2 implies that every strong bisimulation is also a weak one (since, for every $\alpha \in Act$, $\xrightarrow{\alpha} \subseteq \xRightarrow{\widehat{\alpha}}$). This yields the desired connection to LTS equivalence: for every $P, Q \in Prc$,

$$LTS(P) = LTS(Q) \implies P \sim Q \implies P \approx Q.$$

Furthermore trace equivalence is implied if the definition is adapted:

$$P \approx Q \implies \hat{Tr}(P) = \hat{Tr}(Q)$$

where $\hat{Tr}(P) := \{\widehat{w} \mid w \in Tr(P)\} \subseteq (N \cup \overline{N})^*$.

Properties of Weak Bisimulation III

Another important property is

Lemma 8.5

For every $P \in Prc$,

$$P \approx \tau.P$$

Proof.

We show that

$$\rho := \{(P, \tau.P)\} \cup id_{Prc}$$

is a weak bisimulation:

1. let $P \xrightarrow{\alpha} P'$
   $\implies \tau.P \xrightarrow{\tau} P \xrightarrow{\alpha} P'$
   $\implies \tau.P \stackrel{\widehat{\alpha}}{\Longrightarrow} P'$ with $P' \rho P'$ (since $id_{Prc} \subseteq \rho$)
2. the only transition of $\tau.P$ is $\tau.P \xrightarrow{\tau} P$;
   it is simulated by $P \stackrel{\varepsilon}{\Longrightarrow} P$ with $P \rho P$

□

Properties of Weak Bisimulation III

Another important property is

Lemma 8.5

For every $P \in Prc$,

$$P \approx \tau.P$$

Proof.

We show that

$$\rho := \{(P, \tau.P)\} \cup id_{Prc}$$

is a weak bisimulation:

1. let $P \xrightarrow{\alpha} P'$
   $\Longrightarrow \tau.P \xrightarrow{\tau} P \xrightarrow{\alpha} P'$
   $\Longrightarrow \tau.P \stackrel{\widehat{\alpha}}{\Longrightarrow} P'$ with $P' \rho P'$ (since $id_{Prc} \subseteq \rho$)
2. the only transition of $\tau.P$ is $\tau.P \xrightarrow{\tau} P$;
   it is simulated by $P \stackrel{\varepsilon}{\Longrightarrow} P$ with $P \rho P$

□

Properties of Weak Bisimulation III

Another important property is

Lemma 8.5

For every $P \in Prc$,

$$P \approx \tau.P$$

Proof.

We show that

$$\rho := \{(P, \tau.P)\} \cup id_{Prc}$$

is a weak bisimulation:

1. let $P \xrightarrow{\alpha} P'$
   $\implies \tau.P \xrightarrow{\tau} P \xrightarrow{\alpha} P'$
   $\implies \tau.P \stackrel{\widehat{\alpha}}{\Longrightarrow} P'$ with $P' \rho P'$ (since $id_{Prc} \subseteq \rho$)
2. the only transition of $\tau.P$ is $\tau.P \xrightarrow{\tau} P$;
   it is simulated by $P \stackrel{\varepsilon}{\Longrightarrow} P$ with $P \rho P$

□

Properties of Weak Bisimulation III

Another important property is

Lemma 8.5

For every $P \in Prc$,

$$P \approx \tau.P$$

Proof.

We show that

$$\rho := \{(P, \tau.P)\} \cup id_{Prc}$$

is a weak bisimulation:

1. let $P \xrightarrow{\alpha} P'$
   $\implies \tau.P \xrightarrow{\tau} P \xrightarrow{\alpha} P'$
   $\implies \tau.P \overset{\widehat{\alpha}}{\Longrightarrow} P'$ with $P' \rho P'$ (since $id_{Prc} \subseteq \rho$)
2. the only transition of $\tau.P$ is $\tau.P \xrightarrow{\tau} P$;
   it is simulated by $P \overset{\varepsilon}{\Longrightarrow} P$ with $P \rho P$

□

Properties of Weak Bisimulation IV

Using Lemma 8.5, however, we can show that $\approx$ is not a congruence:

It is true that	$b.\mathsf{nil} \approx \tau.b.\mathsf{nil}$	(Lemma 8.5)
but	$a.\mathsf{nil} + b.\mathsf{nil} \not\approx a.\mathsf{nil} + \tau.b.\mathsf{nil}$	(Example 8.3(2))

The other operators are uncritical, i.e., weak bisimilarity is preserved under prefixing, parallel composition, and restriction.

Properties of Weak Bisimulation IV

Using Lemma 8.5, however, we can show that $\approx$ is not a congruence:

It is true that $b.\mathsf{nil} \approx \tau.b.\mathsf{nil}$ (Lemma 8.5)
but $a.\mathsf{nil} + b.\mathsf{nil} \not\approx a.\mathsf{nil} + \tau.b.\mathsf{nil}$ (Example 8.3(2))

The other operators are uncritical, i.e., weak bisimilarity is preserved under prefixing, parallel composition, and restriction.

Properties of Weak Bisimulation IV

Using Lemma 8.5, however, we can show that $\approx$ is not a congruence:

It is true that	$b.\mathsf{nil} \approx \tau.b.\mathsf{nil}$	(Lemma 8.5)
but	$a.\mathsf{nil} + b.\mathsf{nil} \not\approx a.\mathsf{nil} + \tau.b.\mathsf{nil}$	(Example 8.3(2))

The other operators are uncritical, i.e., weak bisimilarity is preserved under prefixing, parallel composition, and restriction.

Properties of Weak Bisimulation IV

Using Lemma 8.5, however, we can show that $\approx$ is not a congruence:

It is true that	$b.\mathsf{nil} \approx \tau.b.\mathsf{nil}$	(Lemma 8.5)
but	$a.\mathsf{nil} + b.\mathsf{nil} \not\approx a.\mathsf{nil} + \tau.b.\mathsf{nil}$	(Example 8.3(2))

The other operators are uncritical, i.e., weak bisimilarity is preserved under prefixing, parallel composition, and restriction.

Properties of Weak Bisimulation V

However deadlock sensitivity is guaranteed if τ-actions are appropriately handled:

Theorem 8.6

Let $P, Q \in Prc$ such that $P \approx Q$. Then, for every $w \in (N \cup \overline{N})^*$,

$$P \stackrel{w}{\Longrightarrow} \not\longrightarrow \quad \iff \quad Q \stackrel{w}{\Longrightarrow} \not\longrightarrow .$$

Proof.

analogously to Theorem 6.3 (induction on $|w|$) □

Properties of Weak Bisimulation V

However deadlock sensitivity is guaranteed if τ-actions are appropriately handled:

Theorem 8.6

Let $P, Q \in Prc$ such that $P \approx Q$. Then, for every $w \in (N \cup \overline{N})^*$,

$$P \stackrel{w}{\Longrightarrow}\not\longrightarrow \quad \iff \quad Q \stackrel{w}{\Longrightarrow}\not\longrightarrow .$$

Proof.

analogously to Theorem 6.3 (induction on $|w|$) □

Moreover we have:

Lemma 8.7

For every $P, Q, R \in Prc$,

1. $P + Q \approx Q + P$
2. $P + (Q + R) \approx (P + Q) + R$
3. $P + \mathsf{nil} \approx P$
4. $P \parallel Q \approx Q \parallel P$
5. $P \parallel (Q \parallel R) \approx (P \parallel Q) \parallel R$
6. $P \parallel \mathsf{nil} \approx P$

Proof.

similar to Lemma 6.1 (strong bisimulation; omitted) □

Properties of Weak Bisimulation VI

Moreover we have:

Lemma 8.7

For every $P, Q, R \in Prc$,

1. $P + Q \approx Q + P$
2. $P + (Q + R) \approx (P + Q) + R$
3. $P + \mathsf{nil} \approx P$
4. $P \parallel Q \approx Q \parallel P$
5. $P \parallel (Q \parallel R) \approx (P \parallel Q) \parallel R$
6. $P \parallel \mathsf{nil} \approx P$

Proof.

similar to Lemma 6.1 (strong bisimulation; omitted) □