Modeling Concurrent and Probabilistic Systems

Lecture 8: Weak Bisimulation

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Outline

Definition (Strong bisimulation)

A relation $\rho \subseteq Prc \times Prc$ is called a strong bisimulation if $P \rho Q$ implies, for every $\alpha \in Act$,

1. $P \xrightarrow{\alpha} P' \implies$ ex. $Q' \in Prc$ such that $Q \xrightarrow{\alpha} Q'$ and $P' \rho Q'$
2. $Q \xrightarrow{\alpha} Q' \implies$ ex. $P' \in Prc$ such that $P \xrightarrow{\alpha} P'$ and $P' \rho Q'$

$P, Q \in Prc$ are called strongly bisimilar (notation: $P \sim Q$) if there exists a strong bisimulation ρ such that $P \rho Q$.

Repetition: Properties of Strong Bisimulation

Properties of strong bisimulation

1. $\sim$ is an equivalence relation
2. $LTS(P) = LTS(Q) \implies P \sim Q$
3. $P \sim Q \implies Tr(P) = Tr(Q)$
4. $\sim$ is a CCS congruence
5. $\sim$ is deadlock sensitive
6. $\sim$ is decidable for finite-state processes

Corollary

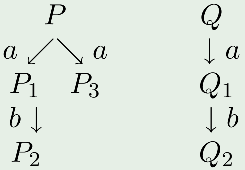
If $P \sim Q$, then Q strongly simulates P, and P strongly simulates Q.

Proof.

A strong bisimulation $\rho \subseteq Prc \times Prc$ for $P \sim Q$ is a strong simulation for both directions. □

Caveat: the converse does generally not hold!

Example



Q simulates P and vice versa, but $P \not\sim Q$

Outline

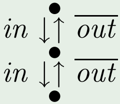
Observation: requirement of exact matching sometimes too strong

Example 8.1

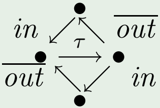
Sequential and parallel two-place buffer:

$$
\begin{aligned}
B_0(in, out) &= in.B_1(in, out)\\
B_1(in, out) &= \overline{out}.B_0(in, out) +\\
& in.B_2(in, out)\\
B_2(in, out) &= \overline{out}.B_1(in, out)
\end{aligned}
$$

$$
\begin{aligned}
B_\parallel(in, out) &= \mathsf{new}\ com\ (B(in, com) \parallel\\
&\qquad B(com, out))\\
B(in, out) &= in.\overline{out}.B(in, out)
\end{aligned}
$$



$\not\sim$



Definition of Weak Bisimulation I

Idea: abstract from silent actions

Definition 8.2

- Given $w \in Act^*$, $\widehat{w} \in (N \cup \overline{N})^*$ denotes the sequence of non-τ-actions in w (in particular, $\widehat{\tau^n} = \varepsilon$ for every $n \in \mathbb{N}$).
- For $w = \alpha_1 \ldots \alpha_n \in Act^*$ and $P, Q \in Prc$, we let

$$P \stackrel{w}{\Longrightarrow} Q \iff P \, (\stackrel{\tau}{\longrightarrow})^* \stackrel{\alpha_1}{\longrightarrow} (\stackrel{\tau}{\longrightarrow})^* \ldots (\stackrel{\tau}{\longrightarrow})^* \stackrel{\alpha_n}{\longrightarrow} (\stackrel{\tau}{\longrightarrow})^* \, Q$$

  (and hence: $\stackrel{\varepsilon}{\Longrightarrow} \; = \; (\stackrel{\tau}{\longrightarrow})^*$).
- A relation $\rho \subseteq Prc \times Prc$ is called a weak bisimulation if $P \rho Q$ implies, for every $\alpha \in Act$,
  1. $P \stackrel{\alpha}{\longrightarrow} P' \implies$ ex. $Q' \in Prc$ such that $Q \stackrel{\widehat{\alpha}}{\Longrightarrow} Q'$ and $P' \rho Q'$
  2. $Q \stackrel{\alpha}{\longrightarrow} Q' \implies$ ex. $P' \in Prc$ such that $P \stackrel{\widehat{\alpha}}{\Longrightarrow} P'$ and $P' \rho Q'$
- $P, Q \in Prc$ are called weakly bisimilar (notation: $P \approx Q$) if there exists a weak bisimulation ρ such that $P \rho Q$.

Definition of Weak Bisimulation II

Remark: each of the two clauses in the definition of weak bisimulation subsumes two cases:

- $P \xrightarrow{\alpha} P'$ where $\alpha \neq \tau$
  $\Longrightarrow$ ex. $Q' \in Prc$ such that $Q\ (\xrightarrow{\tau})^* \xrightarrow{\alpha} (\xrightarrow{\tau})^*\ Q'$ and $P' \rho Q'$
- $P \xrightarrow{\tau} P'$
  $\Longrightarrow$ ex. $Q' \in Prc$ such that $Q\ (\xrightarrow{\tau})^*\ Q'$ and $P' \rho Q'$
  (where $Q' = Q$ is admissible)

Example 8.3

1. Sequential and parallel two-place buffer
2. A counterexample

(on the board)

Outline

Properties of Weak Bisimulation I

Theorem 8.4

$\approx$ is an equivalence relation.

Proof.

in analogy to the corresponding proof for $\sim$ (Theorem 5.2)

In particular, the following characterization is still valid:

$$\approx = \bigcup \{\rho \mid \rho \text{ weak bisimulation}\},$$

i.e., $\approx$ is again itself a weak bisimulation. □

Properties of Weak Bisimulation II

Moreover Definition 8.2 implies that every strong bisimulation is also a weak one (since, for every $\alpha \in Act$, $\xrightarrow{\alpha} \subseteq \xRightarrow{\widehat{\alpha}}$). This yields the desired connection to LTS equivalence: for every $P, Q \in Prc$,

$$LTS(P) = LTS(Q) \implies P \sim Q \implies P \approx Q.$$

Furthermore trace equivalence is implied if the definition is adapted:

$$P \approx Q \implies \hat{Tr}(P) = \hat{Tr}(Q)$$

where $\hat{Tr}(P) := \{\widehat{w} \mid w \in Tr(P)\} \subseteq (N \cup \overline{N})^*$.

Properties of Weak Bisimulation III

Another important property is

Lemma 8.5

For every $P \in Prc$,

$$P \approx \tau.P$$

Proof.

We show that

$$\rho := \{(P, \tau.P)\} \cup id_{Prc}$$

is a weak bisimulation:

1. let $P \xrightarrow{\alpha} P'$
   $\implies \tau.P \xrightarrow{\tau} P \xrightarrow{\alpha} P'$
   $\implies \tau.P \xRightarrow{\widehat{\alpha}} P'$ with $P' \rho P'$ (since $id_{Prc} \subseteq \rho$)
2. the only transition of $\tau.P$ is $\tau.P \xrightarrow{\tau} P$;
   it is simulated by $P \xRightarrow{\varepsilon} P$ with $P \rho P$

□

Properties of Weak Bisimulation IV

Using Lemma 8.5, however, we can show that $\approx$ is not a congruence:

It is true that	$b.\mathsf{nil} \approx \tau.b.\mathsf{nil}$	(Lemma 8.5)
but	$a.\mathsf{nil} + b.\mathsf{nil} \not\approx a.\mathsf{nil} + \tau.b.\mathsf{nil}$	(Example 8.3(2))

The other operators are uncritical, i.e., weak bisimilarity is preserved under prefixing, parallel composition, and restriction.

Properties of Weak Bisimulation V

However deadlock sensitivity is guaranteed if τ-actions are appropriately handled:

Theorem 8.6

Let $P, Q \in Prc$ such that $P \approx Q$. Then, for every $w \in (N \cup \overline{N})^*$,

$$P \stackrel{w}{\Longrightarrow}\not\longrightarrow \quad \Longleftrightarrow \quad Q \stackrel{w}{\Longrightarrow}\not\longrightarrow .$$

Proof.

analogously to Theorem 6.3 (induction on $|w|$) □

Properties of Weak Bisimulation VI

Moreover we have:

Lemma 8.7

For every $P, Q, R \in Prc$,

1. $P + Q \approx Q + P$
2. $P + (Q + R) \approx (P + Q) + R$
3. $P + \mathsf{nil} \approx P$
4. $P \parallel Q \approx Q \parallel P$
5. $P \parallel (Q \parallel R) \approx (P \parallel Q) \parallel R$
6. $P \parallel \mathsf{nil} \approx P$

Proof.

similar to Lemma 6.1 (strong bisimulation; omitted) □