Markovian process algebra

Lecture #22 of Modeling Concurrent and Probabilistic Systems

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July 22, 2008

Overview Lecture #22

⇒ *Markovian process algebra*

- Interactive continuous-time Markov chains
- Markovian (weak) bisimulation revisited
- Sequential probabilistic processes
- Equational laws
- Parallel composition

Motivation

- Performance modeling is an art and requires experience
- Hierachical modeling is complicated

  ⇒ lack of *compositional* specification methods
- Isolation of performance modeling in the design process

  ⇒ need for *integration* with qualitative methods

⇒ Use process algebra for modeling functional *and* quantitative aspects

  ⇒ main benefit: a single consistent system specification for analysis!

Interactive Markov chains

An *interactive Markov chain* is a triple $(S, \textit{Act}, \rightarrow, \mathbf{R}, s_0)$ where

- S is a countable set of states and $s_0 \in S$ is the initial state
- *Act* is a set of actions, and
- $\rightarrow \subseteq S \times \textit{Act} \times S$ is the set of interactive transitions
- $\mathbf{R} \in S \times S \rightarrow \mathbb{R}_{\geqslant 0}$ is the rate function
  - notation: $s \overset{\lambda}{\mapsto} s'$ whenever $\mathbf{R}(s, s') = \lambda > 0$
  - or differently: $\mapsto \subseteq S \times \mathbb{R}_{\geqslant 0} \times S$ is the set of Markovian transitions

each transition system is an IMC and each CTMC is an IMC

Example IMC

On maximal progress

- What is the behaviour in state s with $s \xrightarrow{\alpha}$ and $s \overset{\lambda}{\mapsto}$?
  - if the action α is enabled, no delay takes place in s and α can be performed
  - . . . . . . as the probability of the delay to finish is $1 - e^{-\lambda \cdot 0} = 0$
- How do we know that action α is enabled?
  - we do *not* know this in general, as α may be subject to interaction
  - . . . . . . but in case $\alpha = \tau$, we know that it is always enabled!

$\Rightarrow$ in case $s \xrightarrow{\tau}$ and $s \overset{\lambda}{\mapsto}$, the delay never takes place

  - this is called the *maximal progress assumption*

- Maximal progress becomes apparent in bisimulation and axiomatization

Strong Markovian bisimulation

- Let $(S, \textit{Act}, \rightarrow, \mathbf{R}, s_0)$ be an interactive Markov chain and R an equivalence relation on S

- R is a *Markovian bisimulation* on S if for any $(s,t) \in R$:
  - if $s \xrightarrow{\alpha} s'$ then $\exists t' \in S.\ t \xrightarrow{\alpha} t'$ and $(s',t') \in R$, for all $\alpha \in \textit{Act}$ and
  - if $s \stackrel{\tau}{\not\longrightarrow}$ then $\mathbf{R}(s,C) = \mathbf{R}(t,C)$, for all C in S/R

- s and t are *Markovian bisimilar*, notation $s \sim_m t$, if:

  there exists a Markovian bisimulation R on S with $(s,t) \in R$

Examples

Weak Markovian bisimulation

- Concept: adopt weak bisimulation on immediate actions

$\Rightarrow$ important means to eliminate internal immediate actions

  - essential ingredient to reduce an IMC to a CTMC

- Markovian weak bisimulation:

  - an internal move must be mimicked by a sequence of (0 or more) internal moves
  - an observable move must be mimicked by an obervable move
    . . . . . . that may be preceded and/or followed by a sequence of (0 or more) internal moves
  - the cumulative rate to move to an equivalence class is the same
    . . . . . . in case the state is "stable", i.e., cannot move invisibly

note that Markovian transitions are not combined. Why?

Weak Markovian bisimulation

- Let $(S, \mathit{Act}, \longrightarrow, \mathbf{R}, s_0)$ be an IMC and R an equivalence relation on S

- R is a *weak Markovian bisimulation* on S if for any $(s, t) \in R$:

  if $s \xrightarrow{\tau} s'$ then $\exists t'.\ t \Longrightarrow t'$ and $(s', t') \in R$ and

  if $s \xrightarrow{\alpha} s'$ then $\exists t'.\ t \stackrel{\alpha}{\Longrightarrow} t'$ and $(s', t') \in R$, for all $\alpha \in Act, \alpha \neq \tau$

  and

  if $s \stackrel{\tau}{\not\longrightarrow}$ then $\exists t'.\ t \Longrightarrow t'$ and $t' \stackrel{\tau}{\not\longrightarrow}$ and $\mathbf{R}(s, C) = \mathbf{R}(t', C^\tau)$

- s and t are weak Markovian bisimilar, notation $s \approx_m s'$, if:

  there exists a weak Markovian bisimulation R on S with $(s, s') \in R$

where $C^\tau = \{ s \mid s \Longrightarrow s'$ *and* $s' \in C \}$ *are processes that can invisibly move to* C

Example

A process algebra for sequential processes

A term P in the language tinyMarkovPA is defined as follows:

- nil — nil or stop
- $\alpha.P$ — action prefix
- $(\lambda).P$ — delay prefix
  - behaves as process P after an exponential delay with rate $\lambda \in \mathbb{R}_{\geqslant 0}$
  - i.e., it evolves into P within t time units with probability $1 - e^{-\lambda \cdot t}$
- $P + Q$ — choice
- X — process instantiation
  - for defining equation $X = P$ in the recursive specification E

Operational semantics (I)

The semantics of term P (with recursive specification E) in tinyMPA is given by the IMC

$$(S, \textit{Act}, \longrightarrow, \longmapsto, s_0)$$

with S = all terms in tinyMPA, *Act* $= \alpha(P)$, $s_0 = P$ and $\longrightarrow$ is the smallest relation satisfying:

$$\frac{}{\alpha.P \xrightarrow{\alpha} P} \quad \text{and} \quad \frac{P \xrightarrow{\alpha} P'}{X \xrightarrow{\alpha} P'} \; (X = P \in E)$$

$$\frac{P \xrightarrow{\alpha} P'}{P + Q \xrightarrow{\alpha} P'} \quad \text{and} \quad \frac{Q \xrightarrow{\alpha} Q'}{P + Q \xrightarrow{\alpha} Q'}$$

these are indeed the usual inference rules!

Operational semantics (II)

The Markovian transition relation $\mapsto$ is the smallest relation satisfying:

$$\frac{}{(\lambda).P \overset{\lambda}{\mapsto}_0 P} \quad \text{and} \quad \frac{P \overset{\lambda}{\mapsto}_j P'}{X \overset{\lambda}{\mapsto}_j P'} \, (X = P \in E)$$

$$\frac{P \overset{\lambda}{\mapsto}_j P'}{P + Q \overset{\lambda}{\mapsto}_{1.j} P'} \quad \text{and} \quad \frac{Q \overset{\lambda}{\mapsto}_j Q'}{P + Q \overset{\lambda}{\mapsto}_{2.j} Q'}$$

the reason for having indexed inference rules is the same as for DTMCs

Axiomatization of strong Markovian bisimulation

Axioms for $\sim$

$$\begin{aligned} P + \mathsf{nil} &= P \\ P + Q &= Q + P \\ P + P &= P \\ (P + Q) + R &= P + (Q + R) \end{aligned}$$

Axioms for $\sim_m$

$$\begin{aligned} P + \mathsf{nil} &= P \\ P + Q &= Q + P \\ \alpha.P + \alpha.P &= \alpha.P \\ (P + Q) + R &= P + (Q + R) \\ (\lambda).P + (\mu).P &= (\lambda+\mu).P \\ (\lambda).P + \tau.Q &= \tau.Q \end{aligned}$$

the listed axioms are sound and complete for $\sim_m$

Axiomatization of weak Markovian equivalence

- The τ-laws for $\approx$ on transition systems also hold for $\approx_m$:

$$\begin{aligned} P &= \tau.P \\ M + N + \tau.N &= M + \tau.N \\ M + \alpha.P + \alpha.(\tau.P + N) &= M + \alpha.(\tau.P + N) \end{aligned}$$

- The first axiom implies in particular: $(\lambda).\tau.P = (\lambda).P$
- There is no need for a “delay version” of the second axiom
- Note that the following axiom is *not sound* for $\approx_m$:

$$M + (\lambda).P + (\lambda).(\tau.P + N) = M + (\lambda).(\tau.P + N)$$

Renaming and restriction

Let f : *Act* $\rightarrow$ *Act* be a *renaming* function. The inference rule for $P[f]$ is:

$$\frac{P \xrightarrow{\alpha} P'}{P[f] \xrightarrow{f(\alpha)} P'[f]} \quad \text{and} \quad \frac{P \overset{\lambda}{\mapsto}_j P'}{P[f] \overset{\lambda}{\mapsto}_j P'[f]}$$

For $\beta \in$ *Act*, the derivation rule for restriction new $\beta\ P$ is:

$$\frac{P \xrightarrow{\alpha} P' \quad \alpha \neq \beta}{\text{new } \beta P \xrightarrow{\alpha} \text{new } \beta P'} \quad \text{and} \quad \frac{P \overset{\lambda}{\mapsto}_j P'}{\text{new } \beta P \overset{\lambda}{\mapsto}_j \text{new } \beta P'}$$

Asynchronous parallel composition

For $H \subseteq \mathit{Act}$, the inference rules for $P \parallel_H Q$ are:

$$\frac{P \xrightarrow{\alpha} P'}{P \parallel_H Q \xrightarrow{\alpha} P' \parallel_H Q} \; (\alpha \notin H) \quad \text{and} \quad \frac{Q \xrightarrow{\alpha} Q'}{P \parallel_H Q \xrightarrow{\alpha} P \parallel_H Q'} \; (\alpha \notin H)$$

$$\frac{P \xrightarrow{\alpha} P' \;\wedge\; Q \xrightarrow{\alpha} Q'}{P \parallel_H Q \xrightarrow{\alpha} P' \parallel_H Q'} \; (\alpha \in H)$$

$$\frac{P \overset{\lambda}{\mapsto}_j P'}{P \parallel_H Q \overset{\lambda}{\mapsto}_{(j,0)} P' \parallel_H Q} \quad \text{and} \quad \frac{Q \overset{\lambda}{\mapsto}_j Q'}{P \parallel_H Q \overset{\lambda}{\mapsto}_{(0,j)} P \parallel_H Q'}$$

Justification for parallel composition

Example: an M/M/2/1 queueing system

Expansion law

on the black board

Congruence properties of $\sim_m$

- if $P \sim_m Q$ then $\alpha.P \sim_m \alpha.Q$ for any $\alpha \in$ *Act*
- if $P \sim_m Q$ then $(\lambda).P \sim_m (\lambda).Q$ for any $\lambda \in \mathbb{R}_{>0}$
- if $P \sim_m Q$ then $P + R \sim_m Q + R$ and $R + P \sim_m R + Q$ for any R
- if $P \sim_m Q$ then $P[f] \sim_m Q[f]$ for any f
- if $P \sim_m Q$ then $P \backslash H \sim_m Q \backslash H$ for any H
- if $P \sim_m P'$ and $Q \sim_m Q'$ then $P \parallel_H Q \sim_m P' \parallel_H Q'$ for any H

it can also be proven that $\approx_p$ is a congruence (except for $+$)

Obtaining a CTMC