Stochastic Processes

Lecture #12 of Modeling Concurrent and Probabilistic Processes

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Theme of the course

The theory of modelling and analysis

of concurrent probabilistic systems

Course topics second part

- Stochastic processes
  - discrete-time Markov chains
  - continuous-time Markov chains
- Probabilistic process algebra
  - Operational semantics
  - Behavioural equivalences: simulation and bisimulation
- Probabilities and non-determinism
  - Probabilistic automata
  - Behavioural equivalences
- Case studies

Overview Lecture #12

⇒ *A probability theory refresher*

- Random variables, probability measures etc.
- Stochastic processes
- Memoryless property
- Markov property and stochastic independence

Probability theory is simple, isn't it?

In no other branch of mathematics
is it so easy to make mistakes
as in probability theory

Henk Tijms, "Understanding Probability" (2004)

Measurable space

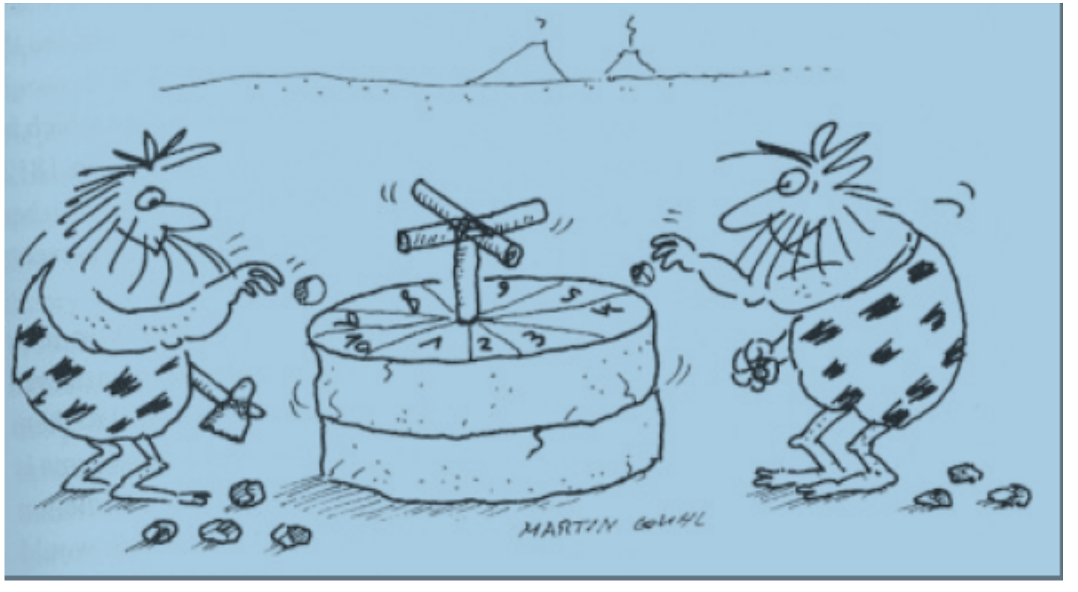
A *sample space* Ω of a chance experiment is a set of elements that have a 1-to-1 relationship to the possible outcomes of the experiment

A *σ-algebra* is a pair $(\Omega, \mathcal{F})$ with $\Omega \neq \varnothing$ and $\mathcal{F} \subseteq 2^{\Omega}$ a collection of subsets of sample space Ω such that:

1. $\Omega \in \mathcal{F}$
2. $A \in \mathcal{F} \quad \Rightarrow \quad \Omega - A \in \mathcal{F}$, and
3. $(\forall i \geqslant 0.\ A_i \in \mathcal{F}) \quad \Rightarrow \quad \bigcup_{i \geqslant 0} A_i \in \mathcal{F}$

The elements of a σ-algebra are called events

Probabilities



Probability space

A *probability space* $\mathcal{P}$ is a structure $(\Omega, \mathcal{F}, \mathrm{Pr})$ with:

- $(\Omega, \mathcal{F})$ is a σ-algebra, and
- $\mathrm{Pr} : \mathcal{F} \to [0, 1]$ is a *probability measure*, i.e.:

1. $\mathrm{Pr}(\Omega) = 1$
2. $\mathrm{Pr}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathrm{Pr}(A_i) \qquad A_i \in \mathcal{F}$ and $A_i \cap A_j = \varnothing$ for $i \neq j$

The elements of a probability space are called *measurable* events

Lemmas in probabilities

- $\Pr(A) = 1 - \Pr(\Omega - A)$
- $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
- $\Pr(A \cap B) = \Pr(A \mid B) \cdot \Pr(B)$
- $A \subseteq B$ implies $\Pr(A) \leqslant \Pr(B)$
- $\Pr(\bigcup_{n \geqslant 1} E_n) = \sup_{n \geqslant 1} \Pr(E_n)$
- $\Pr(\bigcap_{n \geqslant 1} E_n) = \inf_{n \geqslant 1} \Pr(E_n)$

Discrete probability space

- Pr is a *discrete* probability measure on $(\Omega, \mathcal{F})$ if
  - there is a countable set $A \in \Omega$ such that for $a \in A$:

$$\{ a \} \in \mathcal{F} \quad \text{and} \quad \sum_{a \in A} \mathrm{Pr}(\{ a \}) = 1$$

  - e.g., a probability measure on $(\Omega, 2^{\Omega})$
- $(\Omega, \mathcal{F}, \mathrm{Pr})$ is then called a *discrete* probability space
  - otherwise, it is called a continuous probability space
- Examples of discrete probability spaces:
  - throwing a die, number of customers in a shop, . . .

Random variable

Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be measurable spaces

- Function $f : \Omega \to \Omega'$ is a *measurable function* if

$$f^{-1}(A) = \{ a \mid f(a) \in A \} \in \mathcal{F} \quad \text{for all } A \in \mathcal{F}'$$

- Measurable function $X : \Omega \to \mathbb{R}$ is a *random variable*
  - $(\mathbb{R}, \mathrm{B}(\mathbb{R}))$ is the Borel space on the real line
- The *probability distribution* of X is $\mathrm{Pr}_X = \mathrm{Pr} \circ X^{-1}$
  - where Pr is a probability measure on $(\Omega, \mathcal{F})$

Example: rolling a pair of fair dice

Distribution function

The *distribution function* F_X of random variable X is defined by:

$$F_X(d) \;=\; \Pr_X((-\infty, d]) = \Pr(\underbrace{\{\, a \in \Omega \mid X(a) \leqslant d \,\}}_{\{\, X \leqslant d \,\}}) \qquad \text{for real } d$$

Properties:

- F_X is monotonic and right-continuous
- $0 \leqslant F_X(d) \leqslant 1$
- $\lim_{d \to -\infty} F_X(d) = 0$ and
- $\lim_{d \to \infty} F_X(d) = 1$

Discrete / continuous random variables

- The distribution function of a *discrete* random variable can be written as:

$$F_X(d) = \sum_{d_i \leqslant d} f(d_i) \quad \text{with } f \text{ the mass function}$$

- For a continuous random variable:

$$F_X(d) = \int_{-\infty}^{d} f_X(u)\, du \quad \text{with } f \text{ the density function}$$

- F_X is often also called *cumulative density function*

Stochastic process

- *Stochastic process* is a collection of random variables $\{ X_t \mid t \in T \}$
  - casual notation $X(t)$ instead of X_t
  - with all X_t defined on probability space $\mathcal{P}$
  - parameter t (mostly interpreted as “time”) takes values in the set T
- X_t is a random variable whose values are called *states*
  - the set of all possible values of X_t is the state space of the stochastic process
- If the state space is discrete, the stochastic process is discrete and called a *chain*
- Index set T can be discrete/continuous; state space can be discrete or continuous

Classification of stochastic processes (I)

State space	Parameter space T: Discrete	Parameter space T: Continuous
Discrete	DTMC # jobs at k-th job departure	CTMC # jobs at time t
Continuous	waiting time of k-th job	total service time at time t

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Example stochastic processes

- Waiting times of customers in a shop
- Interarrival times of jobs at a production lines
- Service times of a sequence of jobs
- Files sizes that are downloaded via the Internet
- Number of occupied channels in a wireless network
- . . . . . .

Bernouilli process

- Bernouilli *random variable*: $\Pr(X = 1) = p$ and $\Pr(X = 0) = 1-p$
  - moments: $E[X] = p$ and *Var*$[X] = E[X^2] - (E[X])^2 = p{\cdot}(1-p)$
- Bernouilli *process* is a *sequence* of independent and identically distributed Bernouilli r.v.'s $X_1, X_2, \ldots$

Binomial process

- *Binomial* process S_n with $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$
  - probability distribution of "counting process" S_n:

$$\Pr\{ S_n = k \} = \binom{n}{k} p^k \cdot (1-p)^{n-k} \quad \text{for } 0 \leqslant k \leqslant n$$

  - moments: $E[S_n] = n{\cdot}p$ and *Var*$[S_n] = n{\cdot}p{\cdot}(1-p)$

- Let T_i be the number of steps between increments of S_n

$$\Pr\{ T_i = k \} = (1-p)^{k-1}{\cdot}p \quad \text{for } k \geqslant 1$$

$\Rightarrow$ this is a geometric distribution
  - with $E[T_i] = \frac{1}{p}$ and *Var*$[T_i] = \frac{1-p}{p^2}$

Memoryless property

Discrete random variable X is memoryless if:

$\Pr\{\, X = k{+}m \mid X > m \,\} \;=\; \Pr\{\, X = k \,\}$ for any $k \geqslant 1$

any geometrically distributed random variable is memoryless

Joint distribution function

- Stochastic process is a collection of random variables $\{ X_t \mid t \in T \}$
- What is the distribution function of a stochastic process?
- In general, the *joint* distribution function needs to be determined:

$$F_X(d_1, \ldots, d_n; t_1, \ldots, t_n) = \Pr\{ X(t_1) \leqslant d_1, \ldots, X(t_n) \leqslant d_n \}$$

  for all n, $t_1, \ldots, t_n \in T$ and $d_1, \ldots, d_n$

- The structure of F depends on the *stochastic dependency* between the random variables $X(t_i)$

Stochastic independence

Random variables X_i on probability space $\mathcal{P}$ are *independent* if

$$F_X(d_1, \ldots, d_n; t_1, \ldots, t_n) = \prod_{i=1}^{n} F_X(d_i; t_i) = \prod_{i=1}^{n} \Pr\{\, X(t_i) \leqslant d_i \,\}$$

Example independent stochastic process is a *renewal process*

- a discrete-time stochastic process where
- $X(t_1), X(t_2), \ldots$ are independent, identically distributed, non-negative random variables

Minimal possible dependence:

- the next state of the stochastic process only depends on the current state, and not on states assumed previously ($\Rightarrow$ *Markov dependence*)

Markov property

Stochastic process $\{ X(t) \mid t \in T \}$ is a *Markov process* if
for any $t_0 < t_1 < \ldots < t_n < t_{n+1}$:

$$\Pr\{ X(t_{n+1}) \leqslant d_{n+1} \mid X(t_0) = d_0, X(t_1) = d_1, \ldots, X(t_n) = d_n \}$$

$$=$$

$$\Pr\{ X(t_{n+1}) \leqslant d_{n+1} \mid X(t_n) = d_n \}$$

The distribution of $X(t_{n+1})$, given the values $X(t_0)$ through $X(t_n)$, *only depends on the current state* $X(t_n)$

$\Rightarrow$ the history has no influence on the future behaviour

Invariance to time-shifts

Markov process $\{ X(t) \mid t \in T \}$ is *time-homogeneous* iff for any $t' < t$:

$$\Pr\{ X(t) \leqslant d \mid X(t') = d' \} = \Pr\{ X(t - t') \leqslant d \mid X(0) = d' \}$$

the next state only depends on the current state, and
not on how long we have already been in that state