Probabilistic simulation

Lecture #15 of Modeling Concurrent and Probabilistic Systems

Joost-Pieter Katoen and *Thomas Noll*

Lehrstuhl 2: Softwaremodeling and Verification

E-mail: `katoen@cs.rwth-aachen.de`

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Overview Lecture #15

⇒ *Probabilistic simulation*

- Simulation on labeled transition systems
- Weight functions
- Probabilistic simulation (on FPSs)
- Properties of probabilistic simulation

Labeled transition system

A *labeled transition system* *LTS* is a quadruple $(S, \mathit{Act}, \rightarrow, s_0)$ where

- S is a set of states,
- *Act* is a set of actions,
- $\longrightarrow \subseteq S \times \mathit{Act} \times S$ is a transition relation,
- $s_0 \in S$ is the initial state.

Strong simulation

- Let *LTS* $= (S, \mathit{Act}, \rightarrow, s_0)$ and R a binary relation on S
- R is a *strong simulation* on $S \times S$ whenever for $(s, t) \in R$:

  if $s \xrightarrow{\alpha} s'$ then there exists $t' \in S$ such that $t \xrightarrow{\alpha} t'$ and $(s', t') \in R$
- Pictorially:

$$\begin{array}{ccc} s & \xrightarrow{\alpha} & s' \\ R & & \\ t & & \end{array} \quad \text{can be completed to} \quad \begin{array}{ccc} s & \xrightarrow{\alpha} & s' \\ R & & R \\ t & \xrightarrow{\alpha} & t' \end{array}$$

note: transitions of state t do not need to be mimicked by state s

Strong simulation

- State s is *strongly simulated* by t, notation $s \sqsubseteq t$, if:

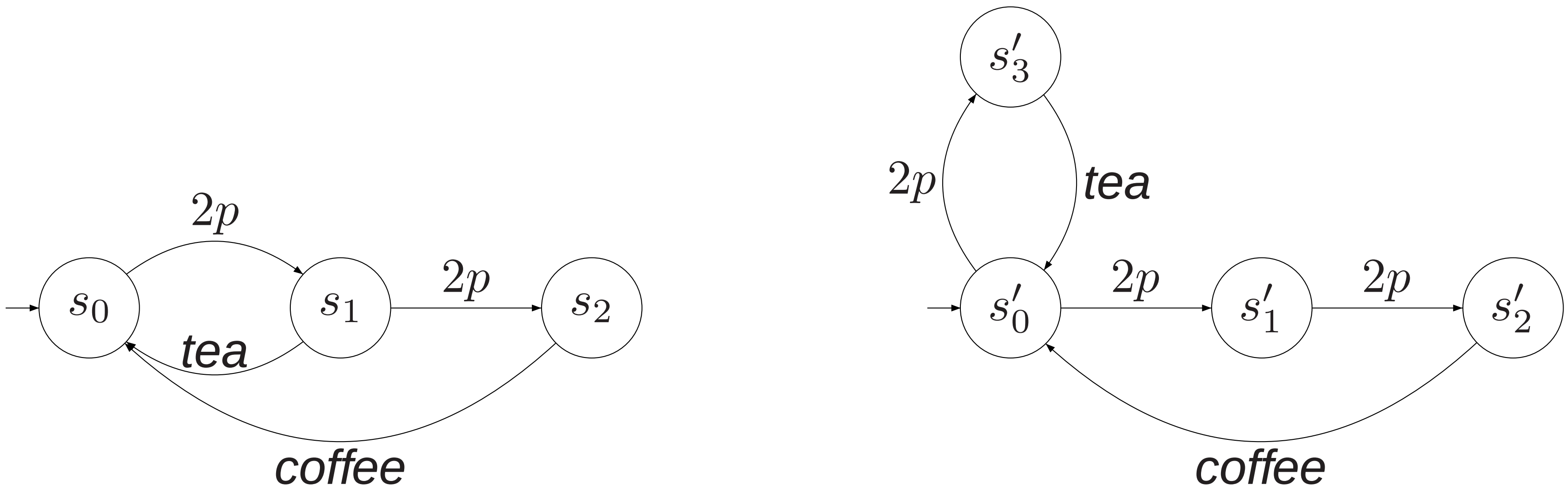
  there exists a strong simulation R such that $(s, t) \in R$

- Property: $\sqsubseteq$ is a pre-order (i.e., reflexive and transitive)

- Lifting $\sqsubseteq$ to transition systems: *LTS* $\sqsubseteq$ *LTS*$'$ if
  - there is a strong simulation R on $S \uplus S'$ such that $(s_0, s_0') \in R$
  - where s_0 and s_0' is the initial state of *LTS* and *LTS*$'$, respectively

- Property: *LTS* $\sqsubseteq$ *LTS*$'$ implies *Traces*(*LTS*) $\subseteq$ *Traces*(*LTS*$'$)
  - the converse holds for if *LTS* is deterministic, but not in general!

Example strong simulation



$\mathit{LTS}_{right} \sqsubseteq \mathit{LTS}_{left}$ but $\mathit{LTS}_{left} \not\sqsubseteq \mathit{LTS}_{right}$ Why?

note that $\mathit{Traces}(\mathit{LTS}_{left}) = \mathit{Traces}(\mathit{LTS}_{right})$

Strong simulation on executions

Whenever we have:

$$\begin{array}{ccccccccc} s_0 & \xrightarrow{\alpha_0} & s_1 & \xrightarrow{\alpha_1} & s_2 & \xrightarrow{\alpha_2} & s_3 & \xrightarrow{\alpha_3} & s_4 \ldots\ldots \\ \sqsubseteq & & & & & & & & \\ t_0 & & & & & & & & \end{array}$$

this can be completed to

$$\begin{array}{ccccccccc} s_0 & \xrightarrow{\alpha_0} & s_1 & \xrightarrow{\alpha_1} & s_2 & \xrightarrow{\alpha_2} & s_3 & \xrightarrow{\alpha_3} & s_4 \ldots\ldots \\ \sqsubseteq & & \sqsubseteq & & \sqsubseteq & & \sqsubseteq & & \sqsubseteq \\ t_0 & \xrightarrow{\alpha_0} & t_1 & \xrightarrow{\alpha_1} & t_2 & \xrightarrow{\alpha_2} & t_3 & \xrightarrow{\alpha_3} & t_4 \ldots\ldots \end{array}$$

Simulation equivalence

- $\sqsubseteq$ is reflexive and transitive, but not necessarily symmetric
  - relations that are reflexive and transitive are also called pre-orders
- The *kernel* of $\sqsubseteq$ is defined by:

$$\simeq \;=\; \sqsubseteq \cap \sqsubseteq^{-1}$$

- Relation $\simeq$ is an equivalence and is called *simulation equivalence*
  - $\mathit{LTS} \simeq \mathit{LTS}'$ iff $\mathit{LTS} \sqsubseteq \mathit{LTS}'$ and $\mathit{LTS}' \sqsubseteq \mathit{LTS}$

$\simeq$ and $\sim$ are slightly different

For strong bisimulation it holds:

$$\begin{array}{ccc} s & \xrightarrow{\alpha} & s' \\ \sim & & \\ t & & \end{array} \quad \text{can be completed to} \quad \begin{array}{ccc} s & \xrightarrow{\alpha} & s' \\ \sim & & \color{red}{\sim} \\ t & \color{red}{\xrightarrow{\alpha}} & \color{red}{t'} \end{array}$$

but for stong simulation equivalence:

$$\begin{array}{ccc} s & & \\ \simeq & & \\ t & \xrightarrow{\alpha} & t' \end{array} \quad \text{can be completed to} \quad \begin{array}{ccc} s & \color{red}{\xrightarrow{\alpha}} & \color{red}{s'} \\ \simeq & & \color{red}{\sqsubseteq} \ (\text{not } \simeq) \\ t & \xrightarrow{\alpha} & t' \end{array}$$

Bisimulation $\neq$ simulation equivalence

Fully probabilistic system

A *fully probabilistic system* (FPS) is a pair $\mathcal{D} = (S, \mathbf{P})$ where:

- S is a countable set of states
- $\mathbf{P} : S \times S \to [0, 1]$ is a *probability matrix* satisfying

$$\sum_{s' \in S} \mathbf{P}(s, s') \in [0, 1] \quad \text{for all} \quad s \in S$$

Deadlocks

- The probability to move from s to (a state in) $C \subseteq S$:

$$\mathbf{P}(s, C) \;=\; \sum_{s' \in C} \mathbf{P}(s, s')$$

- Let $\mathbf{P}(s, \perp) = 1 - \; \mathbf{P}(s, S)$
  - the probability to stay forever in s without performing any transition
  - although $\perp$ is not a "real" state (i.e., $\perp \notin S$), it may be regarded as a *deadlock*
  - $\perp$ is treated in the next lecture as an auxiliary state

s is stochastic iff $\mathbf{P}(s, \perp) = 0$ iff $\mathbf{P}(s, S) = 1$

Discrete-time Markov chain

A DTMC is an FPS where *no* state is sub-stochastic:

$$\mathbf{P}(s, S) = 1 \quad \text{for all} \quad s \in S$$

Probabilistic simulation

- For transition systems, state s' simulates state s if
  - for each successor t of s there is a one-step successor t' of s' that simulates t

$\Rightarrow$ simulation of two states is defined in terms of simulation of successor *states*

- What are successor states in the probabilistic setting?
  - the target of a transition is in fact a probability distribution

$\Rightarrow$ the simulation relation $\sqsubseteq$ needs to be lifted from states to distributions

$\Rightarrow$ probabilistic simulation of two states will be defined in terms of simulation of successor *distributions*

let's first considering lifting an equivalence relation

Distribution functions

- A *distribution* on set S is a function $\mu : S \to [0,1]$ with

$$\sum_{s \in S} \mu(s) \leqslant 1$$

- Distribution μ on S is called *stochastic* if $\mu(\bot) = 0$
  - where $\mu(\bot) = 1 - \sum_{s \in S} \mu(s)$
- Let *Distr*(S) denote the set of all distributions on S
- For $C \subseteq S$, let $\mu(C) = \sum_{s \in C} \mu(s)$

Probabilistic bisimulation

- Let $\mathcal{D} = (S, \mathbf{P})$ be a FPS and R an equivalence relation on S
- R is a *probabilistic bisimulation* on S if for any $(s, s') \in R$:

$$\mathbf{P}(s, C) = \mathbf{P}(s', C) \quad \text{for all} \quad C \text{ in } S/R$$

- s and s' are *probabilistic bisimilar* (or: lumping equivalent), $s \sim_p s'$, if:

  there exists a probabilistic bisimulation R on S with $(s, s') \in R$

it follows that: $s \sim_p s' \quad \Rightarrow \quad \mathbf{P}(s, \bot) = \mathbf{P}(s', \bot)$

Probabilistic bisimulation revisited

- Let $\mathcal{D} = (S, \mathbf{P})$ be a FPS and R an equivalence relation on S
- R is a *probabilistic bisimulation* on S if for any $(s, s') \in R$:

$$\mathbf{P}(s, \cdot) \ \equiv_R \ \mathbf{P}(s', \cdot)$$

where $\equiv_R$ denotes the lifting of R on *Distr*(S) defined by:

$$\mu \ \equiv_R \ \mu' \quad \text{iff} \quad \mu(C) = \mu'(C) \qquad \text{for all} \qquad C \in S/R$$

Probabilistic bisimulation revisited

- Let $\mathcal{D} = (S, \mathbf{P})$ be a FPS and R an equivalence relation on S
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$$\mu \ \equiv_R \ \mu' \quad \text{iff} \quad \mu(C) = \mu'(C) \quad \text{for all} \quad C \in S/R$$

for probabilistic simulation, we replace $\equiv_R$ by a pre-order $\sqsubseteq_R$

this is obtained using the concept of weight functions

Weight function

- Δ *“distributes”* a distribution over X to one over Y
  - such that the total probability assigned by Δ to $y \in Y$
    ... equals the original probability $\mu'(y)$ on Y
  - and symmetrically for the total probability mass of $x \in X$ assigned by Δ
- Δ is *a distribution on $X \times Y$* such that:
  - the probability to select (x, y) with $(x, y) \in R$ is one, and
  - the probability to select $(x, \cdot) \in R$ equals $\mu(x)$, and
  - the probability to select $(\cdot, y) \in R$ equals $\mu'(y)$
- $\perp$ is a “bottom state” that can be simulated by any other state
  - $\Delta(\perp, \perp) > 0$ is possible, but $\Delta(s, \perp) = 0$ for any state s

Weight function

- Let S be a countable set, $R \subseteq S \times S$, and $\mu, \mu' \in \mathit{Distr}(S_\perp)$
- $\Delta \in \mathit{Distr}(S_\perp \times S_\perp)$ is a *weight function* for μ and μ' wrt. R if:

1. $\Delta(s, s') > 0$ implies $(s, s') \in R$ or $s = \perp$
2. $\mu(s) = \sum_{s' \in S_\perp} \Delta(s, s')$ for any $s \in S_\perp$
3. $\mu'(s') = \sum_{s \in S_\perp} \Delta(s, s')$ for any $s' \in S_\perp$

- $\mu \sqsubseteq_R \mu'$ iff there exists a weight function for μ and μ' wrt. R
  - $\sqsubseteq_R$ is the lifting of R (on states) to distributions

Weight function example

- Let $S = \{ s, t, u, v, w \}$ and (sub-stochastic) distribution μ and μ':

$$\mu(s) = \frac{2}{9} \text{ and } \mu(t) = \frac{5}{9} \text{ and } \mu(\cdot) = 0 \text{ otherwise}$$

  and

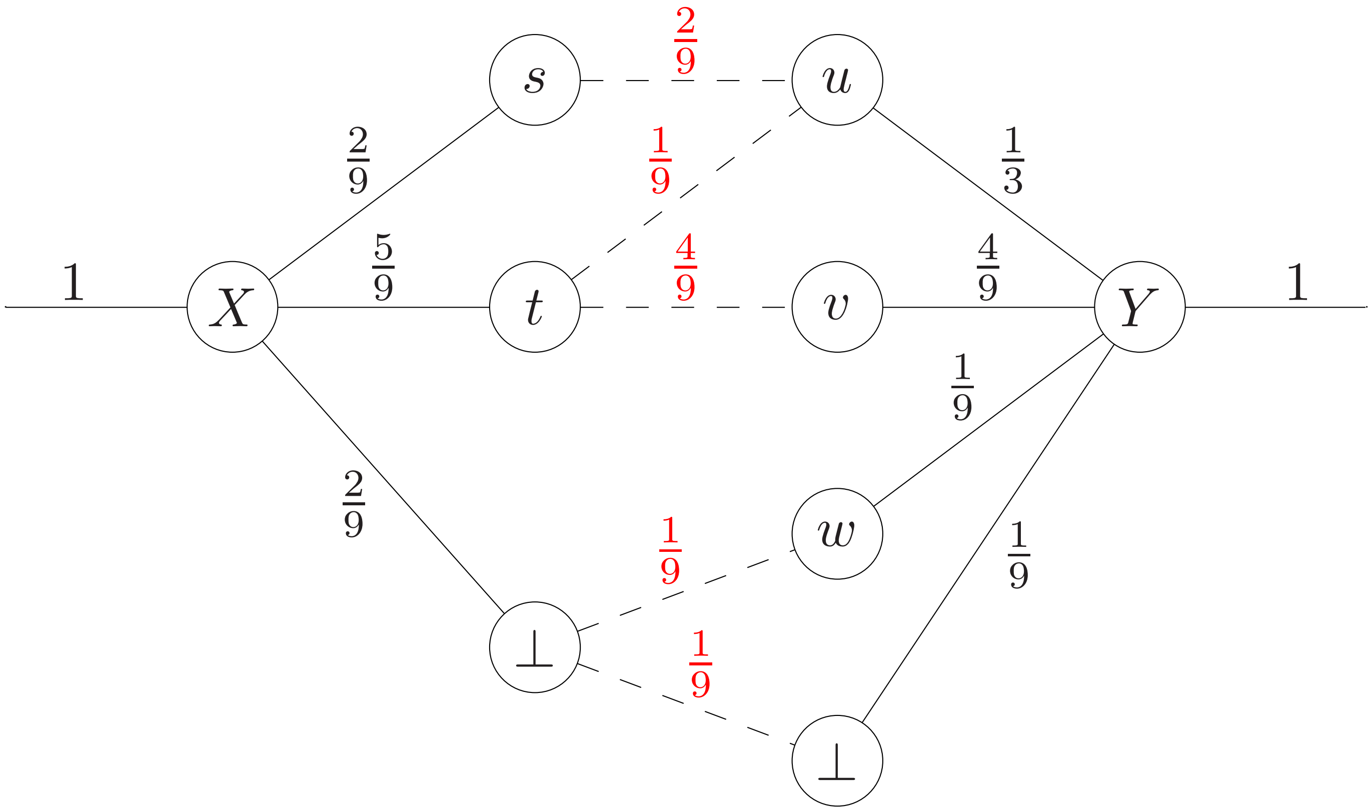
$$\mu'(u) = \frac{1}{3} \text{ and } \mu'(v) = \frac{4}{9} \text{ and } \mu'(w) = \frac{1}{9} \text{ and } \mu'(\cdot) = 0 \text{ otherwise}$$

- For $R = \{ (s, u), (t, u), (t, v) \}$, it follows $\mu \sqsubseteq_R \mu'$, as, e.g., Δ:

$$\Delta(s, u) = \frac{2}{9} \quad \Delta(t, u) = \frac{1}{9} \quad \Delta(t, v) = \frac{4}{9} \quad \Delta(\bot, w) = \frac{1}{9} \quad \Delta(\bot, \bot) = \frac{1}{9}$$

  fulfills the constraints of being a weight function

Weight function example



Properties of weight functions (1)

- Let $\mu_1, \mu_2 \in \textit{Distr}(S)$ and $R_1, R_2 \subseteq S \times S$ with $R_1 \subseteq R_2$. Then:
  - $\mu_1 \sqsubseteq_{R_1} \mu_2$ implies $\mu_1 \sqsubseteq_{R_2} \mu_2$
- Let $\mu_1, \mu_2, \mu_3 \in \textit{Distr}(S)$ and $R_1, R_2 \subseteq S \times S$. Then:
  - $\mu_1 \sqsubseteq_{R_1} \mu_2$ and $\mu_2 \sqsubseteq_{R_2} \mu_3$ implies $\mu_1 \sqsubseteq_{R_1 \odot R_2} \mu_3$
- Let $R \subseteq S \times S$. Then:
  - R is reflexive (transitive) implies $\sqsubseteq_R$ is reflexive (transitive)
- If $R \subseteq S \times S$ is symmetric and $\mu, \mu' \in \textit{Distr}(S)$ with $\mu(S) = \mu'(S)$ then
  - $\mu \sqsubseteq_R \mu'$ iff $\mu' \sqsubseteq_R \mu$

Properties of weight functions (2)

Let $R \subseteq S \times S$ be an equivalence and $\mu, \mu' \in \mathit{Distr}(S)$

- $\mu \equiv_R \mu'$ implies that $\mu \sqsubseteq_R \mu'$ and $\mu(S) = \mu'(S)$
- $\mu(S) = \mu'(S)$ implies that $\mu \equiv_R \mu'$ iff $\mu \sqsubseteq_R \mu'$

$\sqsubseteq_R$ on stochastic distributions is an equivalence and agrees with $\equiv_R$

Probabilistic simulation

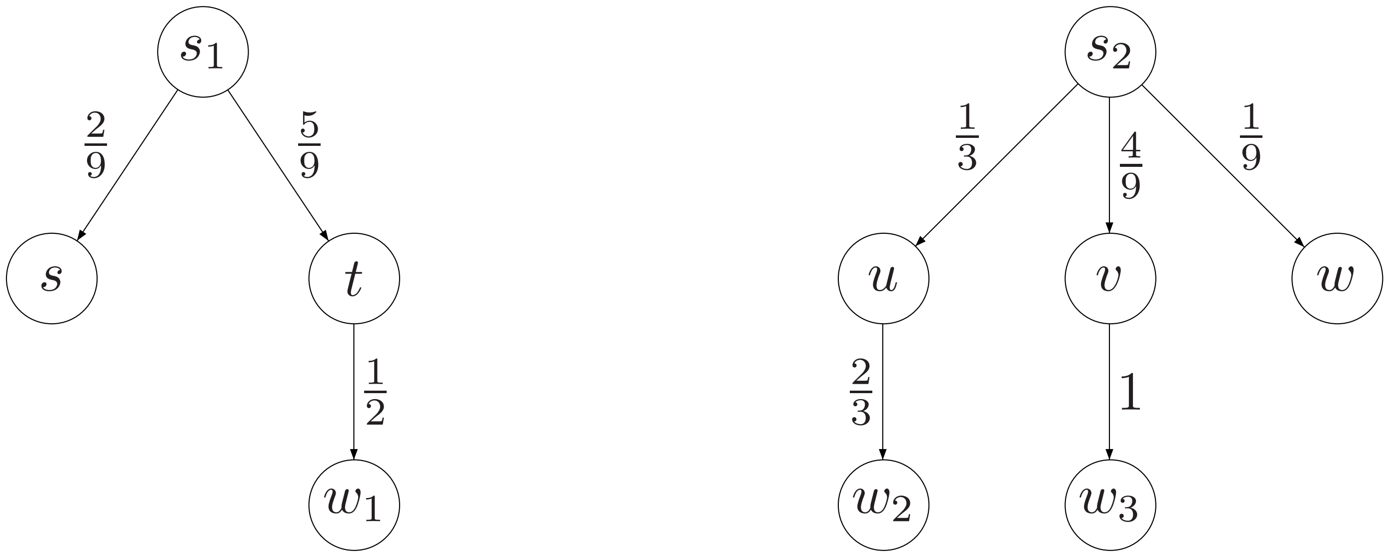
- Let $\mathcal{D} = (S, \mathbf{P})$ be a FPS and $R \subseteq S \times S$
- R is a *strong probabilistic simulation* on S if for all $(s, s') \in R$:

$$\mathbf{P}(s, \cdot) \ \sqsubseteq_R \ \mathbf{P}(s', \cdot)$$

- s' probabilistically simulates s, denoted $s \sqsubseteq_p s'$, if there exists

  a (strong) probabilistic simulation R on S such that $(s, s') \in R$

Probabilistic simulation example



$$R = \{ (s_1, s_2), (s, u), (t, u), (t, v), (w_1, w_2), (w_1, w_3) \}$$

is a probabilistic simulation (cf. weight function on direct successors of s_1 and s_2)

Some properties

1. $\sqsubseteq_p$ is the *coarsest* probabilistic simulation on $\mathcal{D}$
2. $s \sim_p s'$ implies $s \sqsubseteq_p s'$
3. For any DTMC without absorbing states:

   $\sqsubseteq_p$ is symmetric and coincides with $\sim_p$

Upward and downward closure

Let S be a set, $C \subseteq S$, and $R \subseteq S \times S$ be a pre-order Then:

$$\begin{aligned} C\uparrow_R &= \{s' \in S \mid (s,s') \in R \text{ for some } s \in C\} \\ C\downarrow_R &= \{s' \in S \mid (s',s) \in R \text{ for some } s \in C\} \end{aligned}$$

C is called R-*downward-closed* iff $C = C\downarrow_R$

C is called R-*upward-closed* iff $C = C\uparrow_R$

C is R-downward-closed iff $S - C$ is R-upward closed

if R is an equivalence relation, then $s\uparrow_R \;=\; s\downarrow_R \;=\; [s]_R$

Property

For any FPS, $\sqsubseteq_p$ is the coarsest binary relation R on S such that:

$$\text{for all } (s, s') \in R \text{ and } C \subseteq S : \; \mathbf{P}(s, C{\uparrow}_R) \; \leqslant \; \mathbf{P}(s', C{\uparrow}_R)$$

Simulation equivalence = bisimulation

For any FPS:

probabilistic simulation equivalence

coincides with

probabilistic bisimulation

as opposed to transition systems!

Probabilistic reachability

For any $C \subseteq S$ such that $C = C{\uparrow}_{\sqsubseteq_p}$:

$$s \sqsubseteq_p s' \quad \Rightarrow \quad \underbrace{\Pr\left\{s \overset{\leqslant n}{\leadsto} C\right\}}_{p(s,n,C)} \leqslant \Pr\left\{s' \overset{\leqslant n}{\leadsto} C\right\} \qquad \text{for any } n \geqslant 0$$

where

$$p(s,n,C) = \begin{cases} 1 & \text{if } s \in C \\ \sum_{s' \in S} \mathbf{P}(s,s') \cdot p(s', n{-}1, C) & \text{if } s \notin C \text{ and } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

this can be generalized by forbidding paths that visit states in $B \subseteq S$ with $B = B{\uparrow}_{\sqsubseteq_p}$