Bisimulation and Parallel Composition

Lecture #18 of Modeling Concurrent and Probabilistic Systems

Joost-Pieter Katoen and Thomas Noll

Lehrstuhl 2: Softwaremodeling and Verification

E-mail: `katoen@cs.rwth-aachen.de`

July 9, 2009

Overview Lecture #18

⇒ *Bisimulation and parallel composition*

- Process algebra for sequential processes
- Probabilistic bisimulation on PTS
- Congruence properties
- Synchronous parallel composition
- Restriction

A process algebra for sequential processes

The set Prc_p of probabilistic process expressions is defined by the syntax:

- nil (inaction)
- $\alpha.P$ (prefixing)
- $\sum_{j \in J} [p_j] \, P_j$ (probabilistic choice)
  - where J is a finite index set and probability $p_j \in (0, 1)$ with $\sum_{j \in J} p_j = 1$
- $A(\alpha_1, \ldots, \alpha_n)$ (process instantiation)
  - where $A \in Pid$ and $\alpha_i \in$ *Act* $(0 < i \leqslant n)$

there is no nondeterministic choice!

Indexed probabilistic semantics

$$\frac{}{\alpha.P \xrightarrow{\alpha,1}_{0} P} \text{ (Act)}$$

$$\frac{A(\vec{\alpha}) = P \quad P[\vec{\alpha} \mapsto \vec{\beta}] \xrightarrow{\alpha,p}_{j} P'}{A(\vec{\beta}) \xrightarrow{\alpha,p}_{j} P'} \text{ (Call)}$$

$$\frac{P_k \xrightarrow{\alpha,p}_{n} P' \quad k \in J}{\sum_{j \in J} [p_j]P_j \xrightarrow{\alpha,p_k \cdot p}_{k.n} P'} \text{ (Psum)}$$

abbreviate $[p]P + [1-p]Q$ by $P \oplus_p Q$

Recursive equations

Alternatively, let $\to$ be defined as for CCS:

$$\frac{}{\alpha.P \xrightarrow{\alpha} P} \qquad \frac{P \xrightarrow{\alpha} P' \quad A = P}{A \xrightarrow{\alpha} P'} \qquad \frac{P_k \xrightarrow{\alpha} P'}{\sum_{j\in J} [p_k] P_k \xrightarrow{\alpha} P'} \; (k \in J)$$

and define $\mathbf{P}$ as *the least solution* satisfying the recursive equations:

$$\begin{aligned} \mathbf{P}(\alpha.P, \alpha, P) &= 1 \\ \mathbf{P}(\textstyle\sum_{j\in J} [p_j]\, P_j, \alpha, P) &= \displaystyle\sum_{j\in J} p_j \cdot \mathbf{P}(P_j, \alpha, P) \\ \mathbf{P}(A, \alpha, P') &= \mathbf{P}(P, \alpha, P') \quad \text{provided } A = P \text{ and } A \in Pid \end{aligned}$$

Probabilistic bisimulation

Objectives:

- Define probabilistic bisimulation on *Prc*$_p$
  - by lifting the notion of $\sim_p$ on FPS to PTS
- Investigate the basic properties of probabilistic bisimulation
  - e.g., do we have $P \oplus_p$ nil $\sim_p P$ for any $P \in Prc_p$?
- Investigate whether this is a congruence
  - e.g., $P \sim_p Q \;\Rightarrow\; P \oplus_q R \sim_p Q \oplus_q R$ for any $R \in Prc_p$ and $q \in (0,1)$?

Probabilistic transition system

A *probabilistic transition system* (PTS) is a quadruple $(S, \mathit{Act}, \mathbf{P}, s_0)$ where

- S is a countable set of states and $s_0 \in S$ is the initial state
- *Act* is a set of actions, and
- $\mathbf{P} \in S \times \mathit{Act} \times S \rightarrow [0,1]$ a transition probability function satisfying:

$$\sum_{\alpha} \sum_{s' \in S} \mathbf{P}(s, \alpha, s') \in [0,1] \quad \text{for each } s \in S$$

Probabilistic bisimulation

- Let $(S, \mathit{Act}, \mathbf{P}, s_0)$ be a PTS and R an equivalence relation on S
- R is a *probabilistic bisimulation* on S if for any $(s, s') \in R$:

$$\mathbf{P}(s, \alpha, C) = \mathbf{P}(s', \alpha, C) \quad \text{for all} \quad C \text{ in } S/R \text{ and all } \alpha \in \mathit{Act}$$

- s and s' are *probabilistically bisimilar*, notation $s \sim_p s'$, if:

  there exists a probabilistic bisimulation R on S with $(s, s') \in R$

it follows that $s \sim_p s'$ implies $\mathbf{P}(s, \alpha, \perp) = \mathbf{P}(s', \alpha, \perp)$

Alternative definition of $\sim_p$

- Let $(S, \mathit{Act}, \mathbf{P}, s_0)$ be a PTS and R an equivalence relation on S
- R is a *probabilistic bisimulation* on S if for any $(s, s') \in R$ and $\alpha \in \mathit{Act}$:

$$\mathbf{P}(s, \alpha, \cdot) \;\equiv_R\; \mathbf{P}(s', \alpha, \cdot)$$

where $\equiv_R$ denotes the lifting of R on *Distr*(S) defined by:

$$\mu \;\equiv_R\; \mu' \quad \text{iff} \quad \mu(C) = \mu'(C) \qquad \text{for all} \qquad C \in S/R$$

as processes are states, $\sim_p$ is also defined on the set *Prc*$_p$ of processes

Properties of bisimulation (repetition)

$$
\begin{array}{rcll}
P + \mathsf{nil} & \sim & P & \text{(Identity)} \\
P + Q & \sim & Q + P & \text{(Commutativity)} \\
P + P & \sim & P & \text{(Absorption)} \\
(P + Q) + R & \sim & P + (Q + R) & \text{(Associativity)}
\end{array}
$$

Properties of probabilistic bisimulation

$$P \oplus_p \text{nil} \quad \not\sim_p \quad P \qquad \text{(Identity)}$$

$$P \oplus_p Q \quad \sim_p \quad Q \oplus_{1-p} P \qquad \text{(Commutativity)}$$

$$P \oplus_p P \quad \sim_p \quad P \qquad \text{(Absorption)}$$

$$(P \oplus_{p'} Q) \oplus_{q'} R \quad \sim_p \quad P \oplus_p (Q \oplus_q R) \qquad \text{(Associativity)}$$

where $p = p' \cdot q'$, $(1-p) \cdot q = q' \cdot (1-p')$ and $1-q' = (1-p) \cdot (1-q)$

for unguarded recursion: $(A = P \oplus_p A) \; \sim_p \; (A = P)$

Proof

Properties of probabilistic bisimulation (revisited)

$$P \oplus_p \mathsf{nil} \quad \not\sim_p \quad P \qquad \text{(Identity)}$$

$$P \oplus_p Q \quad \sim_p \quad Q \oplus_{1-p} P \qquad \text{(Commutativity)}$$

$$P \oplus_p P \quad \sim_p \quad P \qquad \text{(Absorption)}$$

$$\left(P \oplus_{\frac{p}{p+q}} Q\right) \oplus_{p+q} R \quad \sim_p \quad P \oplus_p \left(Q \oplus_{\frac{q}{1-p}} R\right) \qquad \text{(Associativity)}$$

Congruence properties for $\sim_p$

$\sim_p$ is a *congruence* for prefixing and probabilistic choice:

- $P \sim_p Q \;\Rightarrow\; \alpha.P \sim_p \alpha.Q$ for all α
  - Why? Relation $\{\, (\alpha.P, \alpha.Q) \mid P \sim_p Q \,\} \cup \sim_p$ is a probabilistic bisimulation
- $P \sim_p Q \;\Rightarrow\; P \oplus_p R \sim_p Q \oplus_p R$ for all R
  - Why? For arbitrary R:

$$\{\, (P \oplus_p R, Q \oplus_p R) \mid P \sim_p Q \,\} \cup \sim_p \quad \text{is a probabilistic bisimulation}$$

- $P \sim_p Q \;\Rightarrow\; R \oplus_p P \sim_p R \oplus_p Q$ for all R

recursion requires special treatment and is not considered here

Synchronous parallel composition

Consider just labeled transition systems.

- Let $* : \mathit{Act} \times \mathit{Act} \longrightarrow \mathit{Act}$ map pairs of actions to actions
  - action $\alpha * \beta$ is the simultaneous execution of α and β
- $P \times Q$ denotes the synchronous composition of processes P and Q
- $\times$ is defined by the inference rule: $$\frac{P \xrightarrow{\alpha} P' \;\wedge\; Q \xrightarrow{\beta} Q'}{P \times Q \xrightarrow{\alpha * \beta} P' \times Q'}$$
- Expansion law for $\times$:

$$\sum_{i \in I} \alpha_i . P_i \;\times\; \sum_{j \in J} \beta_j . Q_j \;=\; \sum_{i,j \in I \times J} (\alpha_i * \beta_j).(P_i \times Q_j)$$

Example

$P = a.\mathsf{nil} + b.\mathsf{nil}$ and $Q = \bar{a}.\mathsf{nil} + c.\mathsf{nil}$ and $P \times Q$

Synchronous parallel composition

For $*$: *Act* $\times$ *Act* $\rightarrow$ *Act*, let the derivation rule for $P \times Q$ be:

$$\frac{P \xrightarrow{\alpha,p}_{i} P' \ \wedge \ Q \xrightarrow{\beta,q}_{j} Q'}{P \times Q \xrightarrow{\alpha*\beta,p\cdot q}_{(i,j)} P' \times Q'}$$

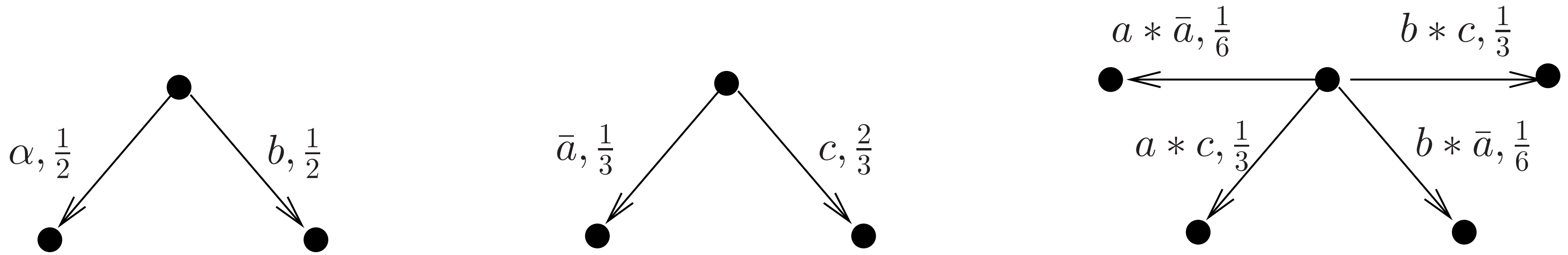
Expansion law for $\times$:

$$\sum_{i \in I} [p_i]\alpha_i.P_i \ \times \sum_{j \in J} [q_j]\beta_j.Q_j \ = \ \sum_{i,j \in I \times J} [p_i \cdot q_j](\alpha_i * \beta_j).(P_i \times Q_j)$$

we obtain the product probability space due to stochastic independence of P and Q

Example of synchronous composition

$P = a.\mathrm{nil} \oplus_{\frac{1}{2}} b.\mathrm{nil}$ and $Q = \bar{a}.\mathrm{nil} \oplus_{\frac{1}{3}} c.\mathrm{nil}$ and $P \times Q$



$\Rightarrow$ PTSs and synchronous composition fit well together

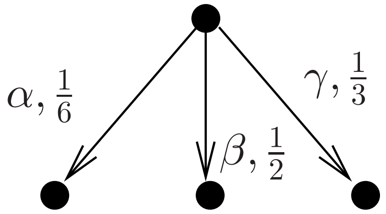
Restriction

- Recall the restriction operator of CCS:
  - new β P declares β as a local name to P
- Formal semantics

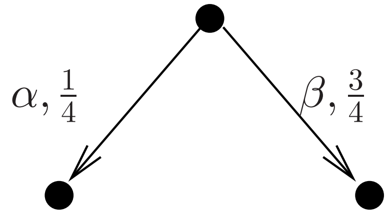
$$\frac{P \xrightarrow{\alpha} P' \quad \alpha \neq \beta}{\text{new } \beta\ P \xrightarrow{\alpha} \text{new } \beta\ P'} \text{ (New)}$$

- What does it mean probabilistically that action β is prohibited?

Restriction: an example



$$P = [\tfrac{1}{6}]\alpha.\mathsf{nil} + [\tfrac{1}{2}]\beta.\mathsf{nil} + [\tfrac{1}{3}]\gamma.\mathsf{nil}$$



$$\mathsf{new}\ \gamma\ P$$

How can the result of restriction be justified?

Justification

- The probabilities in new $\beta\ P$ are conditioned to not performing β
- These probabilities are normalised
  - the normalisation factor = probability that P does not perform β
- Normalisation can be seen as a repeated experiment:
  - probabilistically select one of the alternative transitions
  - in case a prohibited transition (i.e., β) has been selected, start over
  - continue this process until a possible transition (i.e., non-β) has been selected

Semantics of restriction

For $\beta \in Act$, the derivation rule for restriction new $\beta\ P$ is:

$$\frac{P \xrightarrow{\alpha, p}_j P' \quad \alpha \neq \beta}{\text{new } \beta\ P \xrightarrow{\alpha, \frac{p}{\nu(P,\beta)}}_j \text{new } \beta\ P'} \text{ (New)}$$

where

$$\nu(P, \beta) = 1 - \sum_j \{\!|\ p \mid P \xrightarrow{\beta, p}_j P'\ |\!\}$$

is the probability that P does not perform a β-transition

$\{\!|\ \ldots\ |\!\}$ denotes a bag, or a multiset

Asynchronous parallel composition

Recall the derivation rules for CCS:

$$\frac{P \xrightarrow{\alpha} P'}{P \parallel Q \xrightarrow{\alpha} P' \parallel Q} \qquad \frac{Q \xrightarrow{\alpha} Q'}{P \parallel Q \xrightarrow{\alpha} P \parallel Q'} \qquad \frac{P \xrightarrow{\lambda} P' \quad Q \xrightarrow{\bar{\lambda}} Q'}{P \parallel Q \xrightarrow{\tau} P' \parallel Q'}$$

how can these rules be adapted to the probabilistic case?

example on the black board for probabilistic case

Example

Asynchronous parallel composition

- PTSs are closed under synchronous parallel composition
  - synchronous parallel composition can be defined in a rather straightforward way
- PTS are not closed under asynchronous parallel composition
  - as order of autonomous transitions (by P and Q) is not quantified

$\Rightarrow$ Nondeterminism is needed

  - . . . but is not present in the model of PTS

- A more general model is needed: probabilistic automata