Probabilistic Automata

Lecture #19 of Modeling Concurrent and Probabilistic Systems

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Overview Lecture #22

$\Rightarrow$ *Probabilistic automata*

- Probabilistic automata
- Probabilistic (bi)simulation
- Probabilistic process algebra revisited

The importance of nondeterminism

- Implementation freedom as a specification
  - describes *what* the system should do, not *how* it must be implemented
- Scheduling freedom
  - no info about relative speeds of components
- External environment
  - do not stipulate how the environment will behave
- Incomplete information

"There is nothing mysterious about nondeterminism, it arises from the deliberated decision to ignore the factors which influence the selection"

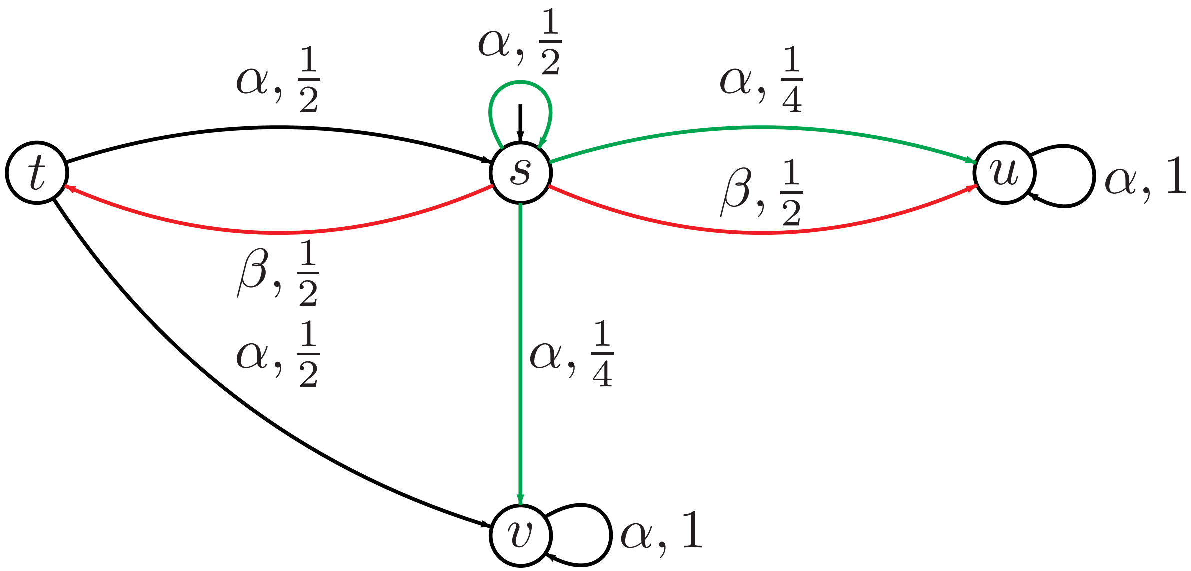
Probabilistic automata

A *probabilistic automaton* (PA) is a quadruple $(S, \mathit{Act}, \rightarrow, s_0)$ where

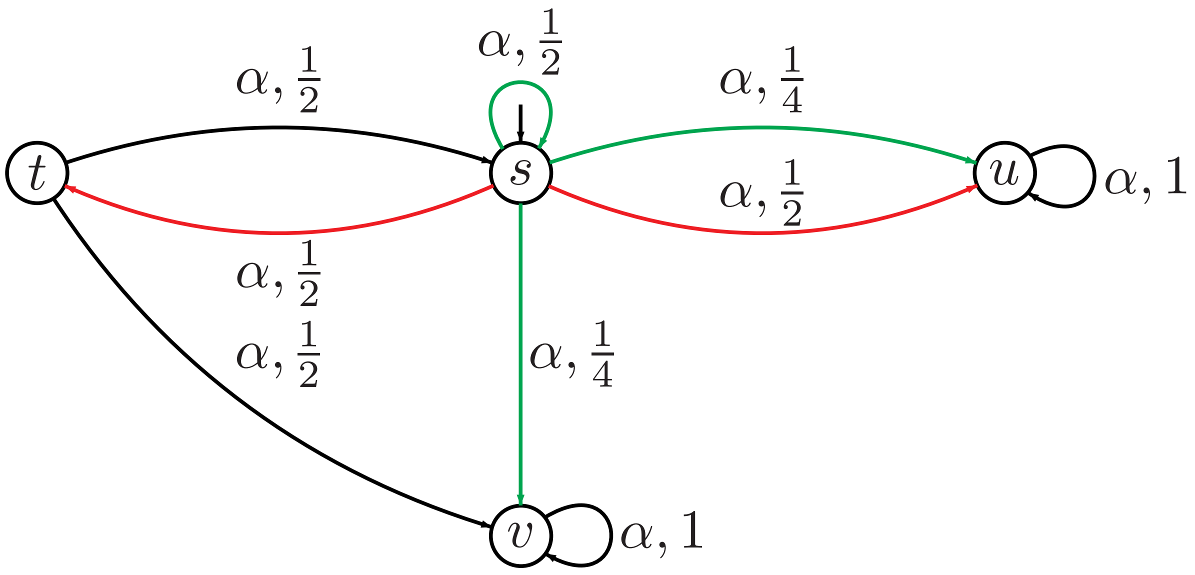
- S is a countable set of states
- *Act* is a countable set of actions, and
- $\rightarrow \subseteq S \times \mathit{Act} \times \mathit{Distr}(S)$ is a transition relation
- $s_0 \in S$ is the initial state

notation: $(s, \alpha, \mu) \in \rightarrow$ is written as $s \xrightarrow{\alpha} \mu$
set of steps in state s is $\mathit{Steps}(s) = \{ (\alpha, \mu) \mid s \xrightarrow{\alpha} \mu \}$

An example PA



Another example PA



Lossy communication channels

on the black board

Reactive probabilistic automata

- Probabilistic automaton $(S, \mathit{Act}, \longrightarrow, s_0)$ is called *reactive* whenever

$$\left(s \xrightarrow{\alpha} \mu \quad \text{and} \quad s \xrightarrow{\alpha} \nu\right) \quad \Rightarrow \quad \mu = \nu$$

- The induced transition probability function $\mathbf{P} : S \times \mathit{Act} \times S \rightarrow [0,1]$ is given by:

$$\mathbf{P}(s, \alpha, s') = \begin{cases} \mu(s') & \text{if } s \xrightarrow{\alpha} \mu \\ 0 & \text{otherwise} \end{cases}$$

- A PA is *non-probabilistic* if for any s: $s \xrightarrow{\alpha} \mu \;\Rightarrow\; \mu(s') = 1$ for some s'

reactive PTSs and TSs are thus subsumed by probabilistic automata

Paths

- A *path* of a PA is an alternating, finite or infinite, sequence

$$s_0 \xrightarrow{\alpha_1, \mu_1} s_1 \xrightarrow{\alpha_2, \mu_2} s_2 \xrightarrow{\alpha_3, \mu_3} s_3 \ldots$$

  such that $s_i \xrightarrow{\alpha_{i+1}} \mu_{i+1}$ and $\mu_{i+1}(s_{i+1}) > 0$

- Let *Paths*$(\mathcal{M})$ the set of paths that start in the initial state of PA $\mathcal{M}$
  - *Paths*$_{fin}(\mathcal{M})$ is the set of finite paths that start in s_0
  - a finite path σ ends in a state, denoted $last(\sigma)$

some example paths of our example on the black board

Probabilistic bisimulation

- Let $(S, \mathit{Act}, \rightarrow, s_0)$ be a PA and R an equivalence relation on S
- R is a *probabilistic bisimulation* on S if for any $(s, s') \in R$:

  if $s \xrightarrow{\alpha} \mu$ then $s' \xrightarrow{\alpha} \mu'$ for some μ' and $\mu \equiv_R \mu'$

- s and s' are *probabilistic bisimilar*, notation $s \sim_p s'$, if:

  there exists a probabilistic bisimulation R on S with $(s, s') \in R$

Probabilistic simulation

- Let $(S, \mathit{Act}, \rightarrow, s_0)$ be a PA and R a binary relation on S
- R is a *probabilistic simulation* on S if for any $(s, s') \in R$:

$$\text{if } s \xrightarrow{\alpha} \mu \text{ then } s' \xrightarrow{\alpha} \mu' \text{ for some } \mu' \text{ and } \mu \sqsubseteq_R \mu'$$

- s' probabilistically simulates s, notation $s \sqsubseteq_p s'$, if:

  there exists a probabilistic simulation R on S with $(s, s') \in R$

Examples

Combined transitions

- $s \overset{\alpha}{\mapsto} \mu$ is a *combined* transition if
  - μ is a *convex combination* of the distributions

$$\textit{Distr}(s, \alpha) = \{\, \mu_i \mid s \xrightarrow{\alpha} \mu_i \,\}$$

  - this means: for each μ_i there is a nonnegative real r_i with

$$\sum_i r_i = 1 \quad \text{and} \quad \mu = \sum_{\mu_i \in \textit{Distr}(s,\alpha)} r_i \cdot \mu_i$$

- Combined transitions := *convex combinations* of existing transitions

Combined bisimulation

- Let $(S, \mathit{Act}, \rightarrow)$ be a PA and R an equivalence relation on S
- R is a *combined* probabilistic bisimulation on S if for any $(s, s') \in R$:

$$\text{if } s \xrightarrow{\alpha} \mu \text{ then } \underbrace{s' \overset{\alpha}{\mapsto} \mu'}_{\text{combined transition}} \text{ for some } \mu' \text{ and } \mu \equiv_R \mu'$$

- s and s' are combined probabilistic bisimilar, notation $s \sim_{cp} s'$, if:

  $\exists$ a combined probabilistic bisimulation R on S with $(s, s') \in R$

in a similar way, $\sqsubseteq_{cp}$ can be defined

Example

Properties

- $\sqsubseteq_p$ is a pre-order
- $\sim_{cp}$ is coarser than $\sim_p$:

$$s \sim_p s' \;\Rightarrow\; s \sim_{cp} s' \qquad \text{but not necessarily the converse}$$

- Simulation equivalence $\simeq_p = \sqsubseteq_p \cap \sqsubseteq_p^{-1}$ is coarser than $\sim_p$:

$$s \sim_p s' \;\Rightarrow\; s \simeq_p s' \qquad \text{but not necessarily the reverse}$$

- For reactive PA, $\simeq_p$ and $\sim_p$ coincide!
  - this is the analogue for deterministic LTSs where $\sim$ and $\simeq$ coincide

Probabilistic process algebra revisited

The set *Prc*$_p$ of probabilistic process expressions is defined by the syntax:

- nil (inaction)
- $P + P$ (choice)
- $\alpha.\left(\sum_{j \in J} [p_j]\, P_j\right)$ (probabilistic choice)
  - where J is a finite index set and probability $p_j \in (0,1)$ with $\sum_{j \in J} p_j = 1$
- $A(\alpha_1, \ldots, \alpha_n)$ (process instantiation)
  - where $A \in Pid$ and $\alpha_i \in$ *Act* $(0 < i \leqslant n)$

Operational semantics

The semantics of P is given by a PA $PA(P)$ where $\rightarrow$ is defined by:

$$\frac{P \xrightarrow{\alpha} \mu}{P + Q \xrightarrow{\alpha} \mu} \text{ (Sum1)} \qquad \frac{Q \xrightarrow{\alpha} \mu}{P + Q \xrightarrow{\alpha} \mu} \text{ (Sum2)}$$

$$\frac{A(\vec{\alpha}) = P \quad P[\vec{\alpha} \mapsto \vec{\beta}] \xrightarrow{\alpha} \mu}{A(\vec{\beta}) \xrightarrow{\alpha} \mu} \text{ (Call)}$$

$$\frac{\mu(P) = \sum_{j \in J, P_j = P} p_j}{\alpha.(\sum_{j \in J} [p_j] P_j) \xrightarrow{\alpha} \mu} \text{ (Psum)}$$

Properties of probabilistic bisimulation

$$P + \mathsf{nil} \sim_p P \qquad \text{(Identity)}$$

$$P + Q \sim_p Q + P \qquad \text{(Commutativity)}$$

$$P + P \sim_p P \qquad \text{(Absorption)}$$

$$(P + Q) + R \sim_p P + (Q + R) \qquad \text{(Associativity)}$$

$$\alpha.(P \oplus_p Q) \sim_p \alpha.(Q \oplus_{1-p} P) \qquad \text{(PCommutativity)}$$

$$\alpha.(P \oplus_p P) \sim_p \alpha.P \qquad \text{(PAbsorption)}$$

$$\alpha.\left(\left(P \oplus_{\frac{p}{p+q}} Q\right) \oplus_{p+q} R\right) \sim_p \alpha.\left(P \oplus_p \left(Q \oplus_{\frac{q}{1-p}} R\right)\right) \qquad \text{(PAssociativity)}$$

Restriction

For $\beta \in$ *Act*, the derivation rule for restriction new $\beta\ P$ is:

$$\frac{P \xrightarrow{\alpha} \mu \qquad \alpha \neq \beta}{\text{new } \beta\ P \xrightarrow{\alpha} \nu} \text{ (New)}$$

where $\nu(\text{new } \beta\ P) = \mu(P)$

Asynchronous CSP-like parallel composition

performing autonomous actions

$$\frac{P \xrightarrow{\alpha} \mu \quad \alpha \notin H}{P \parallel_H Q \xrightarrow{\alpha} \nu} \text{ (Par1)} \quad \text{where} \quad \begin{array}{rcl} \nu(P' \parallel_H Q) & = & \mu(P') \\ \nu(\text{new } H\ Q) & = & \mu(\text{nil}) \end{array}$$

and, symmetrically

$$\frac{Q \xrightarrow{\alpha} \mu \quad \alpha \notin H}{P \parallel_H Q \xrightarrow{\alpha} \nu} \text{ (Par2)} \quad \text{where} \quad \begin{array}{rcl} \nu(P \parallel_H Q') & = & \mu(Q') \\ \nu(\text{new } H\ P) & = & \mu(\text{nil}) \end{array}$$

where new $H\ P$ with $H = \{\, \alpha_1, \ldots, \alpha_n \,\}$ equals new $\alpha_1\ \ldots$ new $\alpha_n\ P$

Asynchronous parallel composition

performing synchronisations

$$\frac{P \xrightarrow{\alpha} \mu \quad \text{and} \quad Q \xrightarrow{\alpha} \mu'}{P \parallel_H Q \xrightarrow{\alpha} \nu} \quad (\alpha \in H)$$

where

$$\nu(P' \parallel_H Q') \quad = \quad \mu(P') \cdot \mu'(Q')$$

parallel composition amounts to the product probability spaces

Example

Expansion law

on the black board

Congruence properties

- if $P_j \sim_p Q_j$ for all $j \in J$ then $\alpha.\left(\sum_{j\in J}[p_j]\,P_j\right) \sim_p \alpha.\left(\sum_{j\in J}[p_j]\,Q_j\right)$ for any $\alpha \in$ *Act*
- if $P \sim_p Q$ then $P + R \sim_p Q + R$ and $R + P \sim_p R + Q$ for any R
- if $P \sim_p Q$ then new α $P \sim_p$ new α Q for any $\alpha \in$ *Act*
- if $P \sim_p P'$ and $Q \sim_p Q'$ then $P \parallel_H Q \sim_p P' \parallel_H Q'$ for any H

it can also be proven that $\sqsubseteq_p$ is a pre-congruence