Software Modeling and Verification
Lehrstuhl für Informatik 2
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Modeling Concurrent and Probabilistic Systems

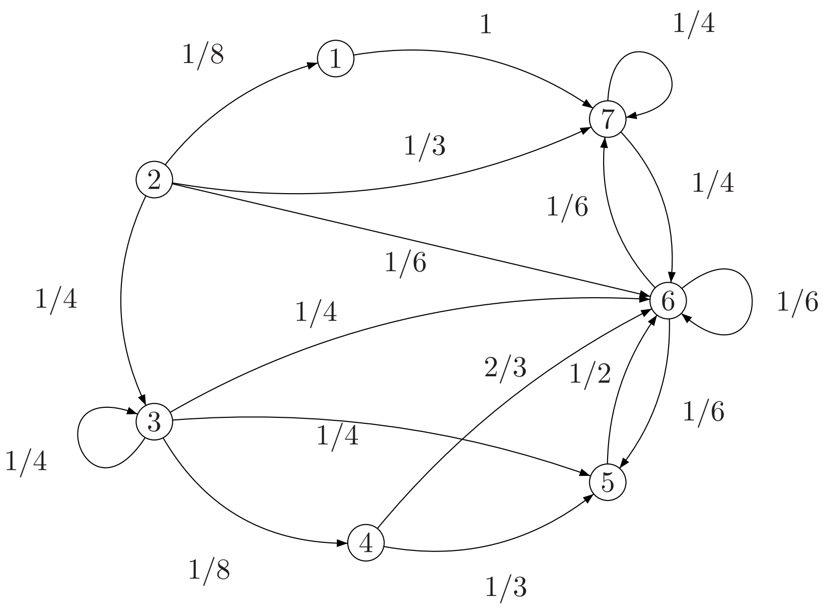
Summer Term 09

– Series 7 –

Hand in until July 1 before the exercise class.

Exercise 1 **(3 points)**

Consider the FPS $\mathcal{D}$, which is given by



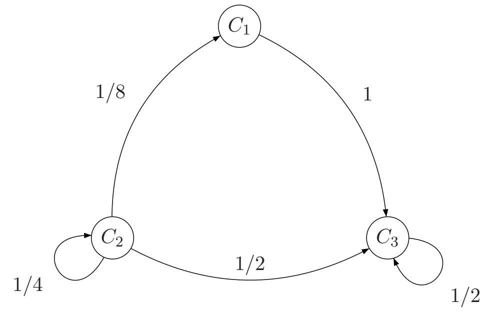
a) Determine $\mathcal{D}/\sim_p$.

b) For each $C \in S/\sim_p$, compute $\underline{p}'_C(3)$, given the initial distribution $\underline{p}(0) = (1/5, 0, 2/15, 1/3, 1/6, 1/9, 1/18)$.

c) Compute the 3-step transient probability distribution in $\mathcal{D}/\sim_p$, given the same initial distribution as in b).

Solution

a) There are three equivalence classes in this FPS. $C_1 = \{1, 4\}$, $C_2 = \{2, 3\}$ and $C_3 = \{5, 6, 7\}$. They form the quotient FPS under $\sim_p$ ($\mathcal{D}/\sim_p$) as follows:



b) In order to do the computation, we first make the FPS to be a DTMC by adding an extra state $S_\perp$. The 1-step transition probability matrix of $\mathcal{D}$ is then becomes

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1/8 & 0 & 1/4 & 0 & 0 & 1/6 & 1/3 & 1/8 \\ 0 & 0 & 1/4 & 1/8 & 1/4 & 1/4 & 0 & 1/8 \\ 0 & 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The 3-step transition probability distribution of $\mathcal{D}$ is

$$\mathbf{P}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1/24 & 5/48 & 5/48 & 3/4 \\ 0 & 0 & 1/64 & 1/128 & 95/1728 & 293/1728 & 19/216 & 85/128 \\ 0 & 0 & 1/64 & 1/128 & 5/64 & 11/64 & 1/16 & 85/128 \\ 0 & 0 & 0 & 0 & 5/108 & 7/54 & 2/27 & 3/4 \\ 0 & 0 & 0 & 0 & 1/72 & 11/144 & 5/144 & 7/8 \\ 0 & 0 & 0 & 0 & 11/432 & 49/864 & 37/864 & 7/8 \\ 0 & 0 & 0 & 0 & 5/288 & 37/576 & 25/576 & 7/8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The initial distribution becomes $(1/5, 0, 2/15, 1/3, 1/6, 1/9, 1/18, 0)$.

$$\underline{p}(3) = \underline{p}(0) \times \mathbf{P}^3 = (0, 0, 1/480, 1/960, 3133/77760, 17039/155520, 10391/155520, 749/960).$$

So the 3-step transient probability to C_1 is

$$\begin{aligned} \underline{p}'_{C_1}(3) &= \sum_{s \in C_1} \underline{p}_s(3) \\ &= \underline{p}_{s_1}(3) + \underline{p}_{s_4}(3) \\ &= 0 + 1/960 \\ &= 1/960 \end{aligned}$$

The 3-step transient probability to C_2 is

$$\begin{aligned} \underline{p}'_{C_2}(3) &= \sum_{s \in C_2} \underline{p}_s(3) \\ &= \underline{p}_{s_2}(3) + \underline{p}_{s_3}(3) \\ &= 0 + 1/480 \\ &= 1/480 \end{aligned}$$

And the 3-step transient probability to C_3 is

$$\begin{aligned} \underline{p}'_{C_3}(3) &= \sum_{s \in C_3} \underline{p}_s(3) \\ &= \underline{p}_{s_5}(3) + \underline{p}_{s_6}(3) + \underline{p}_{s_7}(3) \\ &= 3133/77760 + 17039/155520 + 10391/155520 \\ &= 13/60 \end{aligned}$$

So the 3-step transient distribution computed from $\underline{p}(3)$ is $(1/960, 1/480, 13/60, 749/960)$.

c) Again we extend the quotient FPS with a deadlock state $s_\perp$ to form a DTMC. The initial distribution of the quotient FPS is $\underline{p}'(0) = (8/15, 2/15, 1/3, 0)$.

In $\mathcal{D}/\sim_p$, $\mathbf{P}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1/8 & 1/4 & 1/2 & 1/8 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and $\mathbf{P}'^3 = \begin{pmatrix} 0 & 0 & 1/4 & 3/4 \\ 1/128 & 1/64 & 5/16 & 85/128 \\ 0 & 0 & 1/8 & 7/8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

The initial distribution is $\underline{p}'(0) = (8/15, 2/15, 1/3, 0)$.

The 3-step transient probability distribution in $\mathcal{D}/\sim_p$ is derived by

$$\underline{p}'(0) \times \mathbf{P}'^3 = (1/960, 1/480, 13/60, 749/960),$$

which is equivalent to the result in b).

□

Exercise 2 (5 points)

Let $\mathcal{D} = (S, \mathbf{P})$ be a DTMC and $T \subseteq S, L \subseteq S$. For $s \in S$, let

$$\mathit{Prob}(s, L, T) = \Pr\{X(i) \in T \text{ for some } i \geq 0 \text{ and } X(j) \in L \text{ for all } 0 \leq j < i \mid X(0) = s\}$$

a) Give a recurrent equation for $\mathit{Prob}(s, L, T)$.

b) Now **let** $L \in 2^{S/\sim_p}$ **and** $T \in S/\sim_p$. Show that $s \sim_p s'$ implies $\mathit{Prob}(s, L, T) = \mathit{Prob}(s', L, T)$.

Solution

a)

$$\mathit{Prob}(s, L, T) = \begin{cases} 1 & \text{if } s \in T \\ \sum_{s''} \mathbf{P}(s, s'') \cdot \mathit{Prob}(s'', L, T) & \text{if } s \in L, s \notin T \\ 0 & \text{o.w.} \end{cases}$$

b) Distinguish the following cases.

(i) $s \in T$. Trivial, as $s \sim_p s'$ and $T \in S/\sim_p$ implies $s' \in T$. So $\mathit{Prob}(s, L, T) = 1 = \mathit{Prob}(s', L, T)$.

(ii) $s \notin L, s \notin T$. Then also $s' \notin L, s' \notin T$ and $\mathit{Prob}(s, L, T) = 0 = \mathit{Prob}(s', L, T)$.

(iii) $s \in L, s \notin T$. Then the vector $(\mathit{Prob}(s, L, T))_{s \in (S \cap L) \setminus T}$ is the **smallest solution** of the linear equation system of

$$x_s = \sum_{s'' \in (S \cap L) \setminus T} \mathbf{P}(s, s'') \cdot x_{s''} \qquad (*)$$

Now consider the **smallest solution** $(x_B)_{B \in S/\sim_p, B \subseteq (S \cap L) \setminus T}$ of the following linear equation system:

$$x_B = \sum_{C \in S/\sim_p, C \subseteq (S \cap L) \setminus T} \mathbf{P}(s_B, C) \cdot x_C$$

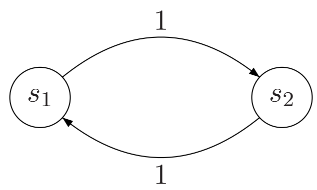
where $s_B \in B$.

Now show that $x_s = x_B$ for any $s \in B \in S/\sim_p$ and $B \subseteq (S \cap L) \setminus T$. This is done by showing that vector $(y_s)_{s\in(S\cap L)\setminus T}$ is a solution to $(*)$ where $y_s = x_B$ if $s \in B$ and $B \in S/\sim_p$ and $B \subseteq (S \cap L) \setminus T$. This goes as follows. Substitute $y_{s''}$ into $(*)$ for $x_{s''}$. This yields:

$$
\begin{array}{rc}
 & x_s = \sum_{s''\in(S\cap L)\setminus T} \mathbf{P}(s,s'')\cdot y_{s''} \\
\Longleftrightarrow & (*\quad (S\cap L)\setminus T \text{ can be partitioned into equivalence classes under } \sim_p \quad *) \\
 & x_s = \sum_{s''\in C, C\in S/\sim_p, C\subseteq(S\cap L)\setminus T} \mathbf{P}(s,s'')\cdot y_{s''} \\
\Longleftrightarrow & (*\quad y_{s''} = x_C \text{ for } s''\in C, C\in S/\sim_p, C\subseteq(S\cap L)\setminus T \quad *) \\
 & x_s = \sum_{s''\in C, C\in S/\sim_p, C\subseteq(S\cap L)\setminus T} \mathbf{P}(s,s'')\cdot x_C \\
\Longleftrightarrow & x_s = \sum_{C\in S/\sim_p, C\subseteq(S\cap L)\setminus T} \sum_{s''\in C} \mathbf{P}(s,s'')\cdot x_C \\
\Longleftrightarrow & (*\quad \text{as } x_C \text{ is constant for all } s''\in C \quad *) \\
 & x_s = \sum_{C\in S/\sim_p, C\subseteq(S\cap L)\setminus T} \underbrace{(\sum_{s''\in C} \mathbf{P}(s,s''))}_{P(s,C)}\cdot x_C \\
\Longleftrightarrow & x_s = \underbrace{\sum_{C\in S/\sim_p, C\subseteq(S\cap L)\setminus T} \mathbf{P}(s,C)\cdot x_C}_{x_B}
\end{array}
$$

So $y_s = x_B = x_s = Prob(s, L, T)$ for any $s \in B, B/\sim_p$. □

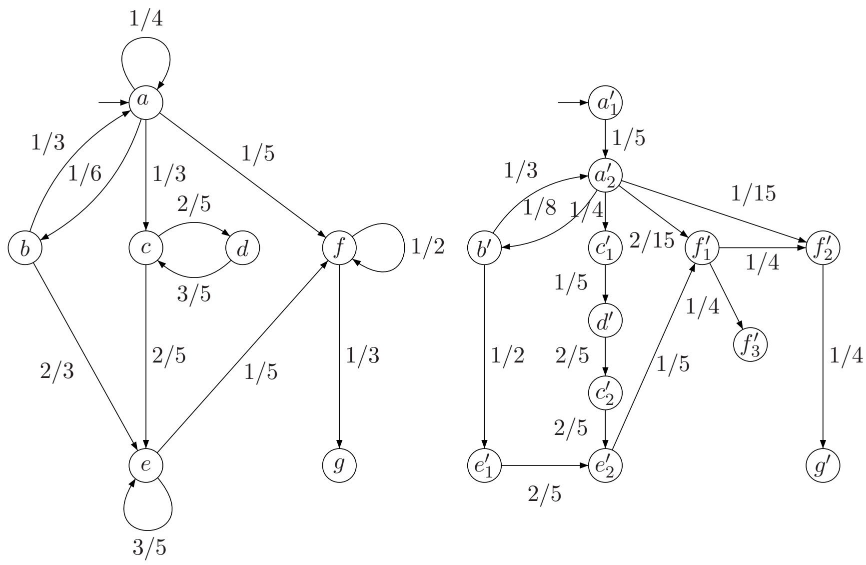
Remark Why the vector $(Prob(s, L, T))_{s\in(S\cap L)\setminus T}$ is the smallest but not the unique solution to $(*)$? Consider the following DTMC where $s_1, s_2 \in L$:



In this setting, $(*)$ is the system of linear equations composed by $x_{s_1} = x_{s_2}$ and $x_{s_2} = x_{s_1}$. Obviously, it has more than 1 solution, where the smallest one $x_{s_1} = x_{s_2} = 0$ is the probability we are looking for. □

Exercise 3 **(2 points)**

Given two FPSs $\mathcal{D}_l$ and $\mathcal{D}_r$ as follows:



Do we have:

a) $(\mathcal{D}_l, a) \sqsubseteq_p (\mathcal{D}_r, a'_1)$?

b) $(\mathcal{D}_r, a'_1) \sqsubseteq_p (\mathcal{D}_l, a)$?

Solution

a) No. Because there does not exist a strong probabilistic simulation which incorperates (a, a'_1). Since the sum of the probabilities of all the outgoing transitions from a is $1/4 + 1/6 + 1/3 + 1/5 = 19/20$, while the sum from a'_1 is $1/5$, which is less than $19/20$. It means that there does not exist a weight function over $\mathbf{P}(a, \cdot)$ and $\mathbf{P}(a'_1, \cdot)$. It also means that a can do more than a'_1, so that a cannot be simulated by a'_1.

b) Yes. $R = \{(a'_1, a), (a'_2, a), (b', b), (c'_1, c), (c'_2, c), (d', d), (e'_1, e), (e'_2, e), (f'_1, f), (f'_2, f), (f'_3, f), (g', g)\}$ is a strong probabilistic bisimulation. We illustrate some of the weight functions, the rest is similar.

- To establish the weight function for $\mathrm{P}(a'_2, \cdot)$ and $\mathrm{P}(a, \cdot)$:
  Since $\mathrm{P}(a'_2, b') = 1/8$,$\mathrm{P}(a'_2, c'_1) = 1/4$,$\mathrm{P}(a'_2, f'_1) = 2/15$ and $\mathrm{P}(a'_2, f'_2) = 1/15$, $\mathrm{P}(a, a) = 1/4$, $\mathrm{P}(a, b) = 1/6$,$\mathrm{P}(a, c) = 1/3$ and $\mathrm{P}(a, f) = 1/5$, the weight function Δ:

$$\Delta(b', b) = 1/8,\ \Delta(c'_1, c) = 1/4,\ \Delta(f'_1, f) = 2/15,\ \Delta(f'_2, f) = 1/15,$$
$$\Delta(\perp, a) = 1/4,\ \Delta(\perp, b) = 1/24,\ \Delta(\perp, c) = 1/12,\ \Delta(\perp, \perp) = 1/20$$

- To establish the weight function for $\mathrm{P}(e'_1, \cdot)$ and $\mathrm{P}(e, \cdot)$:
  Since $\mathrm{P}(e'_1, e'_2) = 2/5$, $\mathrm{P}(e, e) = 3/5$, $\mathrm{P}(e, f) = 1/5$, the weight function Δ':

$$\Delta'(e'_2, e) = 2/5,\ \Delta'(\perp, f) = 1/5,\ \Delta'(\perp, e) = 1/5,\ \Delta'(\perp, \perp) = 1/5$$

- To establish the weight function for $\mathrm{P}(f'_1, \cdot)$ and $\mathrm{P}(f, \cdot)$:
  Since $\mathrm{P}(f'_1, f'_2) = 1/4$, $\mathrm{P}(f'_1, f'_3) = 1/4$, $\mathrm{P}(f, g) = 1/3$, $\mathrm{P}(f, f) = 1/2$, $\mathrm{P}(f, g) = 1/3$, the weight function Δ'':

$$\Delta''(f'_1, f) = 1/4,\ \Delta''(f'_1, f) = 1/4,\ \Delta''(\perp, g) = 1/3,\ \Delta''(\perp, \perp) = 1/6$$

Remark: For the algorithms for deciding strong simulation relation in a DTMC, please refer to Section 4 in the paper "Flow Faster: Efficient Decision Algorithms for Probabilistic Simulations" by Lijun Zhang et al.

□