Compositional Modeling and Minimization of Time-Inhomogeneous Markov Chains

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Outline

1. **Motivation**
   - Continuous-Time Markov Chains
   - Inhomogeneous Continuous-Time Markov Chains
2. **Transient probability distribution**
   - General case
   - Special case
3. **Inhomogeneous Interactive Markov Chains**
4. **Strong bisimulation**
   - ICTMC Strong bisimulation
   - I^2MC Strong bisimulation
   - Quotienting Algorithm
5. **Conclusions and Future work**

ICTMC

Time-dependence

- Current techniques for evaluation of performance and dependability of computer and communications systems assume time-independence.

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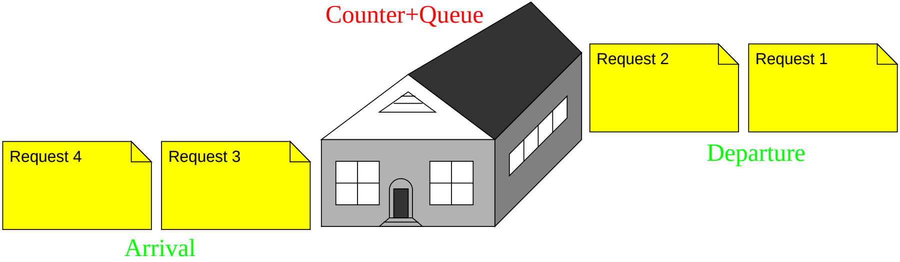
Time-dependence

- The failure of hardware components is time-dependent. Failure rates follow a bath-tub curve.
- Reliability of software in embedded systems is time-dependent.
- The process of battery depletion is time-dependent.

CTMC - (Student canteen)

A counter processing requests:

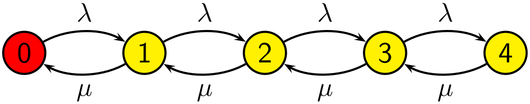


Configuration

- Each request arrives to the counter with rate λ.
- The counter processes the requests with rate μ.
- Additional requests are placed in the queue.

CTMC - 2 (State of the art)

CTMC model of canteen example:



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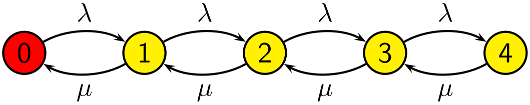

CTMC results

- Transient distribution for CTMCs is well defined.
- Interactive Markov Chain=CTMC+LTS (Labeled Transition Systems).
- Compositional specification of IMCs using Process Algebra.
- Bisimulation technique for state-space minimization.
- A CTL-like logics for property specification on CTMCs.

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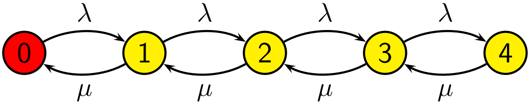

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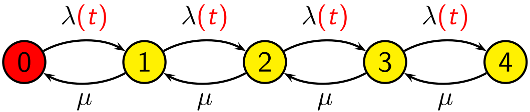
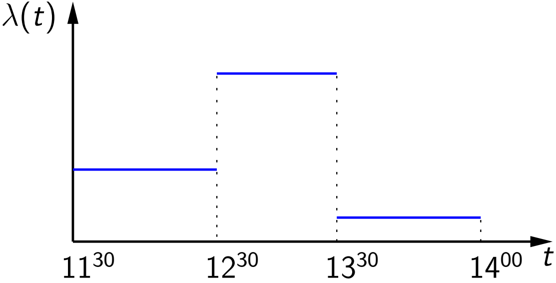
? - is a model with time-varying rates.

CTMC - 3 (State of the art)





ICTMC

Definition

An Inhomogeneous Continuous-Time Markov Chain (ICTMC) is a tuple $\mathcal{C} = (\mathbb{S}, \mathbf{R})$ where:

- $\mathbb{S} = \{1, 2, \ldots, n\}$ is a countable set of states, and
- $\mathbf{R}(t) = [R_{i,j}(t) \geq 0] \in \mathbb{R}_+^{n \times n}$ is a time-dependent rate matrix, where $R_{i,j}(t)$ is the rate between states $i, j \in \mathbb{S}$ at time $t \geq 0$.

- $\mathbf{E}(t) = diag[E_i(t)] \in \mathbb{R}_+^{n \times n}$ is the exit rate diagonal matrix, with $E_i(t) = \sum_{j \in \mathbb{S}} R_{i,j}(t)$ $i, j \in \mathbb{S}$ and $i \neq j$.

ICTMC - 2 (Measures)

- Probability to leave some state s in Δt units of time at time t:

$$\underbrace{1 - e^{-\int_0^{\Delta t} E_s(t+\ell)d\ell}}_{\text{ICTMC}} \qquad \underbrace{1 - e^{-E_s \Delta t}}_{\text{CTMC}}$$

- Probability to select transition $s \rightarrow s'$ at time t:

$$\underbrace{\int_0^{\infty} R_{s,s'}(t+\tau)e^{-\int_0^{\tau} E_s(t+\ell)d\ell}d\tau}_{\text{ICTMC}} \qquad \underbrace{\frac{R_{s,s'}}{E_s}}_{\text{CTMC}}$$

- Probability to make transition $s \rightarrow s'$ in Δt units of time at time t:

$$\underbrace{\int_0^{\Delta t} R_{s,s'}(t+\tau)e^{-\int_0^{\tau} E_s(t+\ell)d\ell}d\tau}_{\text{ICTMC}} \qquad \underbrace{\frac{R_{s,s'}}{E_s}(1 - e^{-E_s \Delta t})}_{\text{CTMC}}$$

ICTMC

Transient distribution - (General case)

Definition

Transient probability distribution - $\Pr\{X(t+\Delta t)=j\}$ denoted by $\pi_j(t+\Delta t)$ is the probability to be in a state j at time $t+\Delta t$:

$$\pi_j(t+\Delta t)=\sum_{i\in\mathbb{S}}\Pr\{X(t)=i\}\cdot\Pr\{X(t+\Delta t)=j|X(t)=i\}$$

Transient probability distribution in matrix form:

$$\boldsymbol{\pi}(t+\Delta t)=\boldsymbol{\pi}(t)\boldsymbol{\Phi}(t+\Delta t,t),$$

- $\boldsymbol{\pi}(t)=[\pi_1(t),\ldots,\pi_n(t)]$ and
- $\boldsymbol{\Phi}(t+\Delta t,t)$ - transition probability matrix.

Transient distribution - 2 (General case)

Transient probability distribution as system of ODEs:

$$\frac{d\boldsymbol{\pi}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\boldsymbol{\pi}(t+\Delta t) - \boldsymbol{\pi}(t)}{\Delta t} = \boldsymbol{\pi}(t) \underbrace{\lim_{\Delta t \to 0} \frac{[\boldsymbol{\Phi}(t+\Delta t, t) - \mathbf{I}]}{\Delta t}}_{\mathbf{Q}(t)}.$$

- Infinitesimal generator $\mathbf{Q}(t)$:

$$\mathbf{Q}(t) = \mathbf{R}(t) - \mathbf{E}(t).$$

Transient distribution - 3 (General case)

- The solution $\pi(t)$:

$$\pi(t) = \pi(t_0)\Phi(t, t_0)$$

- The general form of $\Phi(t, t_0)$ is given by the Peano-Baker series:

$$\Phi(t, t_0) = \mathbf{I} + \int_{t_0}^{t} \mathbf{Q}(\tau_1)d\tau_1 + \int_{t_0}^{t} \mathbf{Q}(\tau_1) \int_{t_0}^{\tau_1} \mathbf{Q}(\tau_2)d\tau_2 d\tau_1 + \dots$$

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- Consider the case of matrix commutativity:

$$\int_{t_0}^{t} \mathbf{Q}(\tau_1) \int_{t_0}^{\tau_1} \mathbf{Q}(\tau_2)d\tau_2 d\tau_1 = \int_{t_0}^{t} \int_{t_0}^{\tau_1} \mathbf{Q}(\tau_2)d\tau_2 \mathbf{Q}(\tau_1)d\tau_1$$

- $\Phi(t, t_0)$ takes the form:

$$\Phi(t, t_0) = e^{\int_{t_0}^{t} \mathbf{Q}(\tau)d\tau}$$

Transient distribution - 4 (Special Case)

Piecewise uniform rate matrix $\mathbf{R}(t)$.

- $M + 1$ - total number of pieces.
- For all $t \in [t_k, t_{k+1})$ and $k \leq M \in \mathbb{N}$:

$$\mathbf{R}(t) = \mathbf{R}_k(t) = f_k(t)\mathbf{R}_k,$$
$$\mathbf{Q}(t) = \mathbf{Q}_k(t) = f_k(t)\mathbf{Q}_k.$$

- Transient probability distribution:

$$\boldsymbol{\pi}(t) = \begin{cases} \boldsymbol{\pi}(t_0)e^{\mathbf{Q}_0 \int_{t_0}^{t} f_0(\tau)d\tau} & \text{, if } t \in [t_0, t_1) \\ \vdots & \vdots \\ \boldsymbol{\pi}(t_M)e^{\mathbf{Q}_M \int_{t_M}^{t} f_M(\tau)d\tau} & \text{, if } t \in [t_M, \infty) \end{cases}$$

- $\boldsymbol{\pi}(t_k) = \boldsymbol{\pi}(t_{k-1})e^{\mathbf{Q}_{k-1} \int_{t_{k-1}}^{t_k} f_{k-1}(\tau)d\tau}$.

ICTMC

$\mathrm{I}^2\mathrm{MC}$

Compositionality

- Modeling large stochastic systems is difficult.
- The solution is to construct models of simpler components.
- ICTMC+LTS used for compositional modeling.
- ICTMC+LTS$=\mathrm{I}^2\mathrm{MC}$.

I^2MC

Compositionality

- Modeling large stochastic systems is difficult.
- The solution is to construct models of simpler components.
- ICTMC+LTS used for compositional modeling.
- ICTMC+LTS=$\mathrm{I^2MC}$.

Definition

An Inhomogeneous Interactive Markov Chain ($\mathrm{I^2MC}$) is a collection $\mathcal{I} = (\mathbb{S}, Act, \rightarrow, \mathbf{R}, s^0)$ where $\mathbb{S}$ and $\mathbf{R}$ are as before,

- Act is a set of actions,
- $\rightarrow \subseteq \mathbb{S} \times Act \times \mathbb{S}$ is a transition relation and
- $s^0 \in \mathbb{S}$ is the initial state.

I^2MC - 2 (Process Algebra)

Grammar

$$P ::= 0 \mid a.P \mid \lambda(t).P \mid P + P \mid P\|_A P \mid P \setminus A \mid X := P$$

Operators:

- Sequential composition - $a.P \xrightarrow{a} P$
- Sequential composition - $\lambda(t).P \xrightarrow{\lambda(t)} P$
- Choice - $P + P$.
- Parallel Composition - $P\|_A P$ (A is the synchronization set).
- Abstraction - $P \setminus A$ (A is the abstraction set).
- Recursion - $X := E[X]$ (E is an expression).

$\mathbf{I^2MC}$ - 3 (SOS rules)

Structural operational semantic (SOS) rules:

$$\frac{}{a.P \xrightarrow{a} P}$$

$$\frac{P \xrightarrow{a} P'}{P+Q \xrightarrow{a} P'}$$

$$\frac{P \xrightarrow{a} P'}{P\|_A Q \xrightarrow{a} P'\|_A Q}(a \notin A)$$

$$\frac{P \xrightarrow{a} P' \quad \text{and} \quad Q \xrightarrow{a} Q'}{P\|_A Q \xrightarrow{a} P'\|_A Q'}(a \in A)$$

$$\frac{P \xrightarrow{a} P'}{P \backslash A \xrightarrow{a} P' \backslash A}(a \notin A)$$

$$\frac{P \xrightarrow{a} P'}{P \backslash A \xrightarrow{\tau} P' \backslash A}(a \in A)$$

$$\frac{}{\lambda(t).P \xrightarrow{\lambda(t)} P}$$

$$\frac{P \xrightarrow{\lambda(t)} P'}{P+Q \xrightarrow{\lambda(t)} P'}$$

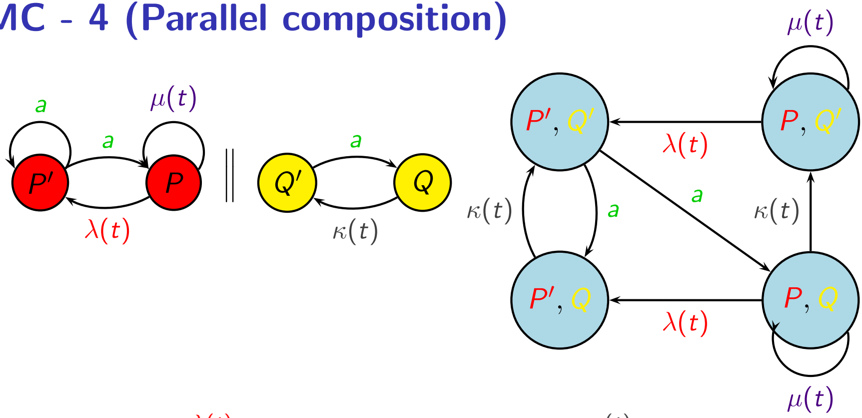
$$\frac{P \xrightarrow{\lambda(t)} P'}{P\|_A Q \xrightarrow{\lambda(t)} P'\|_A Q}$$

$$\frac{P \xrightarrow{\lambda(t)} P'}{P \backslash A \xrightarrow{\lambda(t)} P' \backslash A}$$

$$\frac{E[X:=E/X] \xrightarrow{\lambda(t)} E'}{X:=E \xrightarrow{\lambda(t)} E'}$$

$$\frac{E[X:=E/X] \xrightarrow{a} E'}{X:=E \xrightarrow{a} E'}$$

I^{2}MC - 4 (Parallel composition)



$$\frac{P \xrightarrow{\lambda(t)} P'}{P\|_A Q \xrightarrow{\lambda(t)} P'\|_A Q} \qquad \frac{Q \xrightarrow{\kappa(t)} Q'}{P\|_A Q \xrightarrow{\kappa(t)} P\|_A Q'}$$

Memoryless property

$$Pr\{W(t) \leq t' + \Delta t | W(t) > t'\} = Pr\{W(t + t') \leq \Delta t\}$$

ICTMC

ICTMC Strong bisimulation

Definition

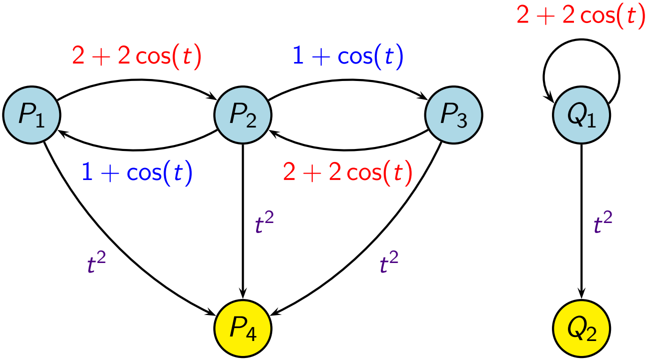
An equivalence relation $\mathcal{R} \subseteq \mathbb{S} \times \mathbb{S}$ is a strong bisimulation whenever: for all $(P, Q) \in \mathcal{R}$, $t \in \mathbb{R}_{\geq 0}$ and $C \in \mathbb{S}/\mathcal{R}$:

$$R(P, C, t) = R(Q, C, t),$$

where

- R is the multiset $R(P, C, t) = \sum_i \{|\lambda(t)|P \xrightarrow{\lambda(t)}_i P', P' \in C|\}$.
- P and Q are strongly bisimilar, denoted $P \sim Q$, if they are contained in some strong bisimulation $\mathcal{R}$, i.e. $(P, Q) \in \mathcal{R}$.

ICTMC Strong bisimulation - 2 (Example)



Equivalence classes:

- $C_1 = \{P_1, P_2, P_3, Q_1\}$
- $C_2 = \{P_4, Q_2\}$

ICTMC Strong bisimulation - 3 (Quotient)

Definition

For the ICTMC $\mathcal{C} = (\mathbb{S}, \mathbf{R}, s^0)$ and $\sim$, the quotient $\mathcal{C}/\sim$ is defined by $\mathcal{C}/\sim = (\mathbb{S}/\sim, \mathbf{R}_\sim, s^0_\sim)$ where $s^0_\sim = [s^0]_\sim$ and $\mathbf{R}_\sim$ is defined by:

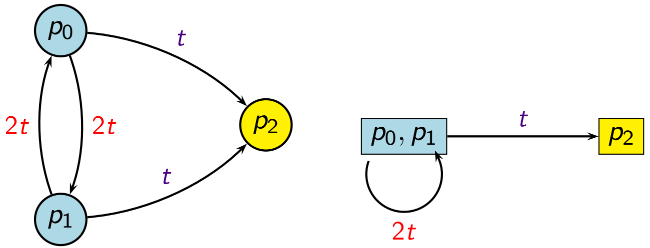
$$R([P]_\sim, [P']_\sim, t) = \frac{\sum_{P'' \in [P]_\sim} R(P'', [P']_\sim, t)}{|[P]_\sim|} \text{ for all } t \in \mathbb{R}_{\geq 0},$$

and $|[P]_\sim|$ the size of $[P]_\sim$.

$$\pi_C(t) = \sum_{s \in C} \pi_s(t) \quad \text{for all } t \in \mathbb{R}_{\geq 0},$$

- $\pi_C(t)$ - transient distribution of state C in quotient $\mathcal{C}/\sim$.

ICTMC Strong bisimulation - 4 (Example)



Equivalence classes:

- $C_1 = \{p_0, p_1\}$
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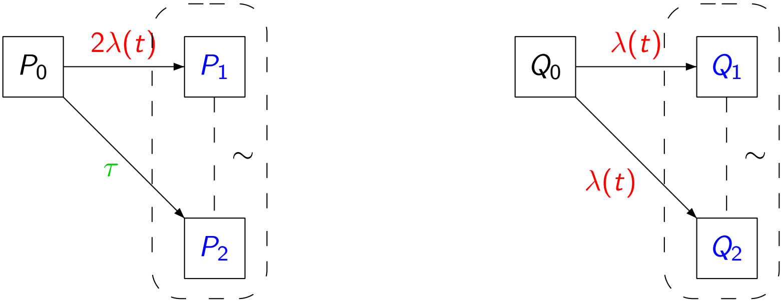
I^2MC Strong bisimulation



Maximal progress

- Internal action τ consumes no time.
- Probability to take $P_0 \xrightarrow{2\lambda(t)} P_1$ in zero time is 0.

I^2MC Strong bisimulation - 2 (Definition)

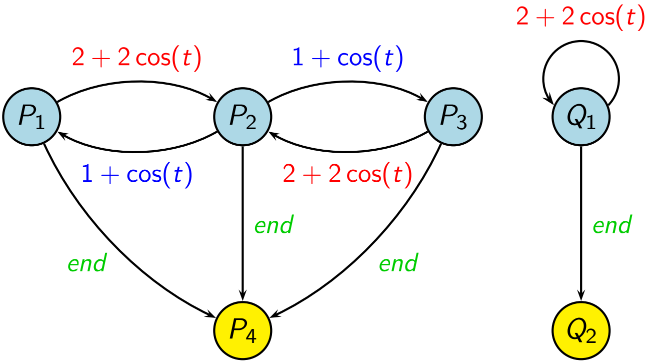
Definition

An equivalence relation $\mathcal{R} \subseteq \mathbb{S} \times \mathbb{S}$ is a strong bisimulation whenever: for all $(P, Q) \in \mathcal{R}$, $t \in \mathbb{R}_{\geq 0}$, $a \in Act$ and $C \in \mathbb{S}/\mathcal{R}$:

- $P \xrightarrow{a} P'$ implies $Q \xrightarrow{a} Q'$ for some Q' and $(P', Q') \in \mathcal{R}$.
- $Q \xrightarrow{a} Q'$ implies $P \xrightarrow{a} P'$ for some P' and $(P', Q') \in \mathcal{R}$.
- $P \overset{\tau}{\nrightarrow}$ (or $Q \overset{\tau}{\nrightarrow}$) implies $R(P, C, t) = R(Q, C, t)$.

P and Q are strongly bisimilar, denoted $P \sim Q$, if they are contained in some strong bisimulation $\mathcal{R}$, i.e. $(P, Q) \in \mathcal{R}$.

I^2MC Strong bisimulation - 3 (Example)



Equivalence classes:

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- $C_2 = \{P_4, Q_2\}$

I^2MC Strong bisimulation - 4 (Congruence)

Congruence properties:

- $P \sim P'$ implies $P \|_A Q \sim P' \|_A Q$.
- $P \sim Q$ implies $a.P \sim a.Q$ for any $a \in Act$.
- $P \sim Q$ implies $\lambda(t).P \sim \lambda(t).Q$ for any $t \in \mathbb{R}_{\geq 0}$.
- $P \sim Q$ implies $P + R \sim Q + R$.
- $P \sim Q$ implies $X := P \sim X := Q$.

I^2MC Strong bisimulation - 5 (Axioms)

Sound and complete axioms for sequential fragment:

- $P + 0 = P$
- $(P + Q) + R = P + (Q + R)$
- $a.P + a.P = a.P$ and $\lambda(t).P + \mu(t).P = (\lambda(t) + \mu(t)).P$
- $\lambda(t).P + \tau.Q = \tau.Q$

Quotienting Algorithm

- Composition modeling results in huge state-spaces.
- Bisimulation minimization achieve exponential state-space reduction for LTS and CTMC.
- We adopt the partition refinement paradigm for minimizing $\mathrm{I^2MC}$.

Quotienting Algorithm

- Composition modeling results in huge state-spaces.
- Bisimulation minimization achieve exponential state-space reduction for LTS and CTMC.
- We adopt the partition refinement paradigm for minimizing $\mathrm{I^2MC}$.

- For general rate function the problem is undecidable.
- We restrict the rate function to piecewise constant, piecewise polynomial and polynomial.

Quotienting Algorithm - 2

Input: $\mathrm{I^2MC}\ (\mathbb{S}, Act, \rightarrow, \mathbf{R})$, M
Output: $\mathrm{I^2MC}\ (\mathbb{S}/\!\sim, Act, \rightarrow_\sim, \tilde{\mathbf{R}})$

$\Pi_\tau := \{s \in \mathbb{S} | s \xrightarrow{\tau}\}$ $\quad \Pi := \{s \in \mathbb{S} | s \overset{\tau}{\nrightarrow}\}$
$L := \mathbf{push}(\{\mathbb{S}\})$
while $L \neq \varnothing$ **do**
$\quad C := \mathbf{pop}(L)$
$\quad [\Pi, L] := Refine_a(\Pi, C, L, Act)$
$\quad [\Pi, L] := Refine(\Pi, C, L, \mathbf{R}, M)$
end while

Rate matrix parameter M:

- $M + 1$ - number of constant or polynomial pieces,
- $M + 1$ - degree of the polynomial for polynomial rate function.

Quotienting Algorithm - 3

Input: $\mathrm{I^2MC}\ (\mathbb{S}, Act, \rightarrow, \mathbf{R}),\ M$
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Refinement functions:

- $Refine_a$ - partition refinement with respect to actions,
- *Refine* - partition refinement with respect to rates.

Refining wrt. actions

Refine$_a$ can be implemented using a simple adaptation of Paige and Tarjan's algorithm.

Complexity

- Time - $\mathcal{O}(m_a \log n)$,
- Space - $\mathcal{O}(m_a)$,

where m_a is the number of action-labelled transitions.

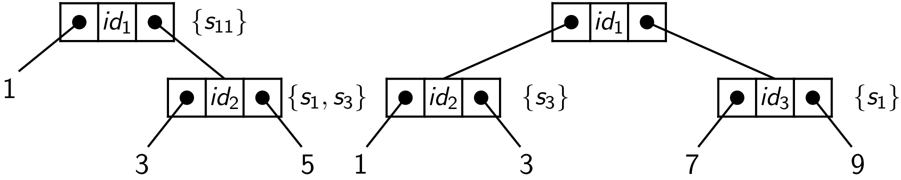
Refining wrt. rates - 2

Input: $Refine(\Pi, C, L, \mathbf{R}, M)$

for $i = 0$ to M **do**
  $B_{\Pi} := \varnothing$
  $\ldots (*initialization*) \ldots$
  for all $s \in Pre(C)$ **do**
    $B' := [s]_{\Pi}$
    delete s *from* B'
    $Insert(\mathrm{T}(B'), s)$
    if $B' \notin B_{\Pi}$ **then**
      $B_{\Pi} := B_{\Pi} \cup B'$
    end if
  end for
  for all $B' \in B_{\Pi}$ **do**
    $\Pi := \Pi \cup$
    $\{\mathrm{T}(B')[1], \ldots, \mathrm{T}(B')[k]\}$
  end for
end for

- $Pre(C)$ - predecessors of C.
- B_{Π} - set of blocks in Π.
- $\mathrm{T}(B')$ - binary tree B'.
- $\mathrm{T}(B')[j]$ - j'th block of $\mathrm{T}(B')$.
- $Insert(\mathrm{T}(B'), s)$ - insert $s \in B'$ into $\mathrm{T}(B')$.

Refining wrt. rates - 3



$T(B')$ node structure:

- *node.left* - points to the left child.
- *node.right* - points to the right child.
- *node.sum* - stores the sum of all rates to splitter C on the i'th piece.
- *node.S* - stores all states with the same *node.sum*.

Refining wrt. rates - 4

Using splay trees instead of balanced binary trees results in:

- Time complexity - $\mathcal{O}(Mm_r \log n)$,
- Space complexity - $\mathcal{O}(m_r)$,

where m_r is the number of rate-labelled transitions.

Refining wrt. rates - 4

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Bisimulation Quotienting Algorithm

- Time complexity - $\mathcal{O}(m_a \log n + M m_r \log n)$,
- Space complexity - $\mathcal{O}(m_a + m_r)$.

ICTMC

Conclusions and Future work

Conclusions:

- Full-fledged process algebra for interactive ICTMCs.
- Congruence results for weak and strong bisimulation.
- Polynomial-time quotienting algorithm.

Future work:

- Extension of quotienting algorithm to a larger class of rate functions.
- Simulation relations for ICTMCs.
- Model-checking algorithms for ICTMCs.

Thank you!