Transition probabilities for Inhomogeneous Continuous Time Markov Chains

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Definition 1 (ICTMC). *An inhomogeneous continuous-time Markov chain is a tuple* $\mathcal{C} = (\mathcal{S}, \boldsymbol{R})$ *where:*

– $S = \{1, 2, \ldots, n\}$ *is a finite set of states.*
– $\boldsymbol{R}(t) = [R_{i,j}(t)] \in \mathbb{R}_+^{\boldsymbol{n}\times\boldsymbol{n}}$ *is a time-dependent rate matrix, with* $i, j \in S$ *and* $t \geq 0$.

Here the exit rate of a state $i \in S$ at time t is $E_i(t) = \sum_{j\in S} R_{i,j}(t)$.

Example 1. Fig. 1 shows an example of a simple queue with three capacities and two servers. The arrival process to the queue is a Poisson process with rate constant λ and the service rate is a function $\mu(t)$ which depends on the global time of the system. Initially the service rate starts at μ_{max} and decreases linearly till μ_{min} at $t = a$. From that moment on, all users are served with constant rate.

An interesting property which can be defined for every ICTMC is the distribution of waiting time in a state. Before that, let us first define the notion of a non-homogeneous Poisson process:

Definition 2 (Inhomogeneous Poisson process). *A stochastic process* $Z : R_{\geq 0} \times \Omega \to S$ *(*Ω *- sample space) is called a non-homogeneous Poisson process with rate* $\lambda(t)$ *if the following relation holds for* $k \in S$*:*

$$Pr\{Z(t) - Z(t_0) = k\} = \frac{\left(\int_{t_0}^{t} \lambda(\tau)d\tau\right)^k e^{-\int_{t_0}^{t} \lambda(\tau)d\tau}}{k!} \tag{1}$$

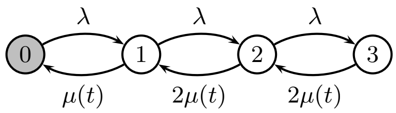
$Z(t) - Z(t_0)$ *- is the number of arrivals in the interval* $[t_0, t]$.

Taking in the above equation $k = 0$ and $t_0 = 0$ we obtain that the probability of no arrivals in the interval $[0, t]$ is $Pr\{Z(t) - Z(0) = 0\} = e^{-\int_0^t \lambda(\tau)d\tau}$. Therefore, we conclude that the probability of no arrivals in the interval $[t, t + \Delta t]$ is:

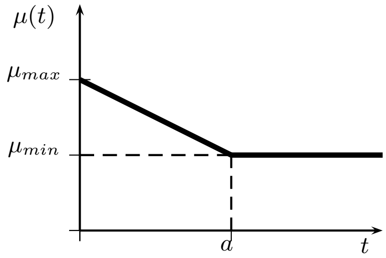
$$Pr\{Z(t + \Delta t) - Z(t) = 0\} = e^{-\int_t^{t+\Delta t} \lambda(\tau)d\tau} = e^{-\int_0^{\Delta t} \lambda(t+\tau)d\tau}$$

Let's take a transition with rate $\lambda(t)$ from some state s to s'. We are interested in the cumulative probability distribution of the firing time of transition λ. For this, we define a random variable $W_\lambda(t)$ whose value at each moment of time t will be the firing time of the transition λ. Then the cumulative probability distribution $Pr\{W_\lambda(t) \leq \Delta t\}$ of the firing time is:

$$Pr\{W_\lambda(t) \leq \Delta t\} = 1 - e^{-\int_0^{\Delta t} \lambda(t+\tau)d\tau} \tag{2}$$



(a) A queue



(b) Service rate $\mu(t)$

Fig. 1. A three state ICTMC.

The above relation can be explained by noting that the probability to have more than one Poisson arrival in interval $[t, t + \Delta t]$ is:

$$1 - Pr\{Z(t + \Delta t) - Z(t) = 0\} = 1 - e^{-\int_0^{\Delta t} \lambda(t+\tau)d\tau}.$$

One interesting characteristic of the above distribution is the memoryless property. This can be proven as follows:

$$\begin{aligned} Pr\{W_\lambda(t) > t' + \Delta t | W_\lambda(t) > t'\} &= \frac{Pr\{W_\lambda(t) > t' + \Delta t, W_\lambda(t) > t'\}}{Pr\{W_\lambda(t) > t'\}} \\ &= \frac{Pr\{W_\lambda(t) > t' + \Delta t\}}{Pr\{W_\lambda(t) > t'\}} \\ &= \frac{e^{-\int_0^{t'+\Delta t} \lambda(t+\tau)d\tau}}{e^{-\int_0^{t'} \lambda(t+\tau)d\tau}} = e^{-\int_{t'}^{t'+\Delta t} \lambda(t+\tau)d\tau} \\ &= Pr\{W_\lambda(t + t') > \Delta t\} \end{aligned}$$

The intuition behind this property is the following. Suppose that at time t transition λ is activated i.e. it starts a clock. Then the probability that transition λ won't fire at time $t' + \Delta t$ given that it didn't fire at time t' ($Pr\{W_\lambda(t) > t' + \Delta t | W_\lambda(t) > t'\}$) is the same as the probability that transition λ won't fire at time Δt given that the transition was activated at time $t + t'$ ($W_\lambda(t + t')$).

Property 1. The cumulative distribution of the waiting time $W_s(t)$ in state s is:

$$Pr\{W_s(t) \leq \Delta t\} = 1 - e^{-\int_0^{\Delta t} E_s(t+\tau)d\tau} \tag{3}$$

where E_s is the exit rate of the state s.

Proof. In order to obtain the distribution of the waiting time in state s we have to consider all transitions that leave this state. Consider all transitions $\lambda_1, \ldots, \lambda_n$ which leave the state s. Then we are interested in the minimum firing time of all these n transitions. More formally this can be stated as $Pr\{W_s(t) \leq \Delta t\}$ with $W_s(t) = min(W_{\lambda_1}(t), \ldots, W_{\lambda_n}(t))$. As all random variables $W_{\lambda_1}(t), \ldots, W_{\lambda_n}(t)$ are independent, we obtain:

$$
\begin{aligned}
Pr\{W_s(t) > \Delta t\} &= Pr\{W_{\lambda_1}(t) > \Delta t\} \cdots Pr\{W_{\lambda_n}(t) > \Delta t\} \\
&= e^{-\int_0^{\Delta t} \lambda_1(t+\tau) d\tau} \cdots e^{-\int_0^{\Delta t} \lambda_n(t+\tau) d\tau} \\
&= e^{-\int_0^{\Delta t} \lambda_1(t+\tau) + \cdots + \lambda_n(t+\tau) d\tau} \\
Pr\{W_s(t) \leq \Delta t\} &= 1 - e^{-\int_0^{\Delta t} \lambda_1(t+\tau) + \cdots + \lambda_n(t+\tau) d\tau}
\end{aligned}
$$

As the exit rate E_s of state s is the sum of the rates of its outgoing transitions, we obtain that:

$$Pr\{W_s(t) \leq \Delta t\} = 1 - e^{-\int_0^{\Delta t} E_s(t+\tau) d\tau}$$

Property 2. The probability $P_{s,s'}(t)$ to take the transition $(s \to s')$ in Δt units of time starting at time t is:

$$P_{s,s'}(t) = \int_0^{\Delta t} \lambda_{s \to s'}(t+\tau) e^{-\int_0^{\tau} E_s(t+\ell) d\ell} d\tau \tag{4}$$

where $\lambda_{s \to s'}$ is the rate of transition $s \to s'$.

Proof. Assume we have n outgoing transitions $\lambda_1, \ldots, \lambda_n$ from state s. We are interested in the probability that some transition i $(s \to s')$ will be selected. More formally, this can be expressed as follows:

$$Pr\{W_{\lambda_i}(t) \text{ is the minimum}\} = Pr\{W_{\lambda_i}(t) < W_{\lambda_j}(t) \text{ for } i \neq j\}$$

For the sake of simplicity we will consider that $\lambda_{j_1}, \ldots, \lambda_{j_{n-1}}$ are the transitions which were not selected.

$$
\begin{aligned}
&Pr\{W_{\lambda_i}(t) < W_{\lambda_j}(t) \text{ for } i \neq j\} = \\
&\int_0^{\Delta t} Pr\{W_{\lambda_i}(t) < W_{\lambda_j}(t) \text{ for } i \neq j | W_{\lambda_i}(t) = \tau\} Pr\{W_{\lambda_i}(t) = \tau\} d\tau = \\
&\int_0^{\Delta t} Pr\{\tau < W_{\lambda_j}(t) \text{ for } i \neq j\} Pr\{W_{\lambda_i}(t) = \tau\} d\tau = \\
&\int_0^{\Delta t} Pr\{W_{\lambda_{j_1}}(t) > \tau\} \cdots Pr\{W_{\lambda_{j_{n-1}}}(t) > \tau\} Pr\{W_{\lambda_i}(t) = \tau\} d\tau
\end{aligned}
$$

As the probability distribution of the firing time of transition λ_i is $Pr\{W_{\lambda_i}(t) = \tau\} = \lambda_i(t+\tau)e^{-\int_0^\tau \lambda_i(t+\ell)d\ell}$ we obtain:

$$\int_0^{\Delta t} Pr\{W_{\lambda_{j_1}}(t) > \tau\} \cdots Pr\{W_{\lambda_{j_{n-1}}}(t) > \tau\} Pr\{W_{\lambda_i}(t) = \tau\} d\tau =$$
$$\int_0^{\Delta t} e^{-\int_0^\tau \lambda_{j_1}(t+\ell)d\ell} \cdots e^{-\int_0^\tau \lambda_{j_{n-1}}(t+\ell)d\ell} \lambda_i(t+\tau)e^{-\int_0^\tau \lambda_i(t+\ell)d\ell} d\tau =$$
$$\int_0^{\Delta t} \lambda_i(t+\tau)e^{-\int_0^\tau \lambda_i(t+\ell)+\sum_{k=1}^{n-1} \lambda_{j_k}(t+\ell)d\ell} d\tau =$$
$$\int_0^{\Delta t} \lambda_i(t+\tau)e^{-\int_0^\tau E_s(t+\ell)d\ell} d\tau$$

Example 2. Consider the ICTMC from Fig. 1. For this chain we observe that the probability of no arrivals in the interval $[0, a]$ for the initial moment of time $t = 0$ is:

$$Pr\{Z(a) - Z(0) = 0\} = e^{-\int_0^a \lambda(\tau)d\tau} = e^{-\lambda \int_0^a d\tau} = e^{-\lambda a}$$

The probability to wait in state 1 for at most a units of time is:

$$Pr\{W_1(0) \leq a\} = 1 - e^{-\int_0^a E_1(\tau)d\tau} = 1 - e^{-\int_0^a \lambda + \mu(\tau)d\tau}$$
$$= 1 - e^{-\lambda a - \frac{a(\mu_{max} + \mu_{min})}{2}}$$

The probability to select transition $1 \xrightarrow{\lambda} 2$ at $t = 0$ is:

$$P_{1,2}(0) = \int_0^\infty \lambda e^{-\int_0^\tau E_s(\ell)d\ell} d\tau = \lambda \int_0^\infty e^{-\int_0^\tau \lambda + \mu(\ell)d\ell} d\tau$$
$$= \lambda \int_0^\infty e^{-\lambda\tau - \frac{\tau(\mu_{max} + \mu_{min})}{2}} d\tau = \frac{2\lambda}{2\lambda + \mu_{max} + \mu_{min}}$$

From the above computations we get that the probability to move from state 1 to state 2 in a units of time starting at time $t = 0$ is:

$$P_{1,2}(0, a) = \frac{2\lambda}{2\lambda + \mu_{max} + \mu_{min}} \left(1 - e^{-a\frac{2\lambda + \mu_{max} + \mu_{min}}{2}}\right)$$