

Bisimulation and Logical Preservation for Continuous-Time Markov Decision Processes

Martin R. Neuhäußer^{1,2} Joost-Pieter Katoen^{1,2}

¹RWTH Aachen University, Germany

²University of Twente, The Netherlands

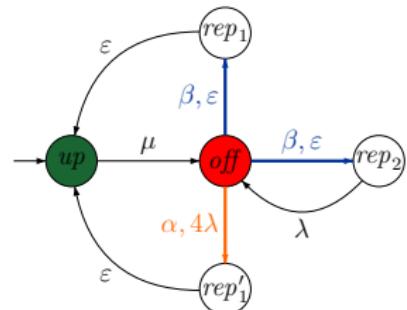
CONCUR 2007, Lisbon, Portugal

Repairman's Roulette

Maintenance of a safety critical system:

- System is operational in state *up*.

Mean delay to next failure: $\frac{1}{\lambda}$
 Probability to fail within time t : $e^{-\lambda t}$



- A failure moves the system from *up* to *off*.

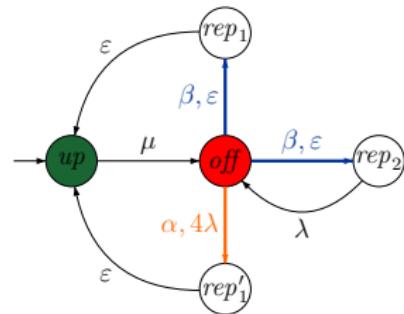
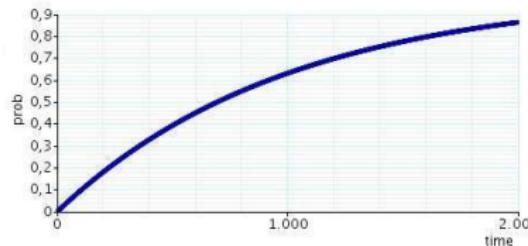
The state of the repairman

Repairman can repair up to 4 failures
 Repairman can repair up to 4 failures

Repairman's Roulette

Maintenance of a safety critical system:

- System is operational in state *up*.
- Failures are exponentially distributed:
 - Mean delay to next failure $\frac{1}{\mu}$.
 - Probability for a failure within time t :



A failure moves the system from up to off.

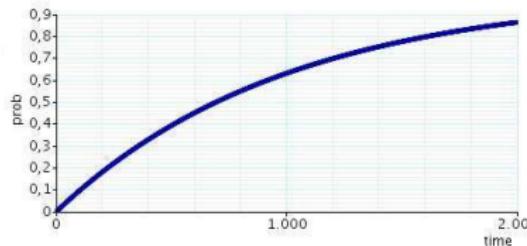
The repairman can choose to repair the system.

Or the repairman can choose to do nothing.

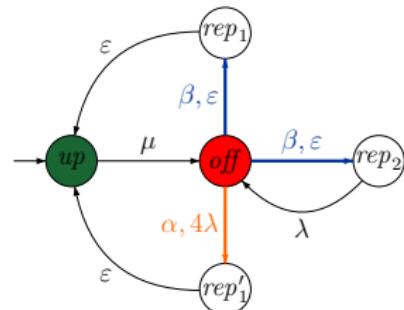
Repairman's Roulette

Maintenance of a safety critical system:

- System is operational in state *up*.
- Failures are exponentially distributed:
 - Mean delay to next failure $\frac{1}{\mu}$.
 - Probability for a failure within time t :



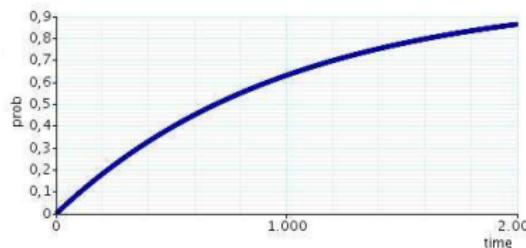
- A failure moves the system from *up* to *off*.



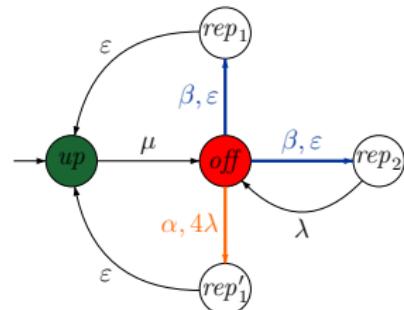
Repairman's Roulette

Maintenance of a safety critical system:

- System is operational in state *up*.
- Failures are exponentially distributed:
 - Mean delay to next failure $\frac{1}{\mu}$.
 - Probability for a failure within time t :



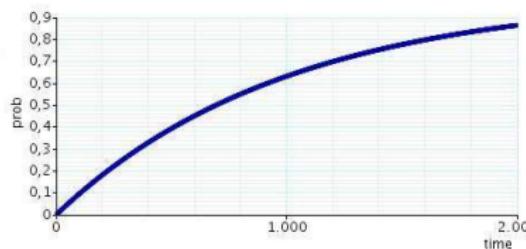
- A failure moves the system from *up* to *off*.
- Two types of repairmen:
 - ① **cautious**: slow and reliable via α or
 - ② **aggressive**: unreliable but fast via β .



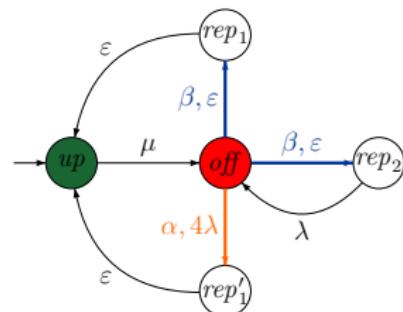
Repairman's Roulette

Maintenance of a safety critical system:

- System is operational in state *up*.
- Failures are exponentially distributed:
 - Mean delay to next failure $\frac{1}{\mu}$.
 - Probability for a failure within time t :



- A failure moves the system from *up* to *off*.
- Two types of repairmen:
 - ① **cautious**: slow and reliable via α or
 - ② **aggressive**: unreliable but fast via β .



Measure of interest: Probability to return to *up* in t time units after a failure.

Why Continuous-Time Markov Decision Processes?

- ① CTMDPs are an important model in
 - stochastic control theory [Qiu et al.]
 - stochastic scheduling [Feinberg et al.]
- ② CTMDPs provide the semantic basis for
 - non-well-specified stochastic activity networks [Sanders et al.]
 - generalised stochastic Petri nets with confusion [Chiola et al.]
 - Markovian process algebras [Hermanns et al., Hillston et al.]

In this talk:

1. Preliminary definitions

2. Semantics of CTMDPs

3. Application of CTMDPs to the analysis of stochastic systems

4. Bisimulation of CTMDPs under strong simulation

Why Continuous-Time Markov Decision Processes?

- ① CTMDPs are an important model in
 - stochastic control theory [Qiu et al.]
 - stochastic scheduling [Feinberg et al.]
- ② CTMDPs provide the semantic basis for
 - non-well-specified stochastic activity networks [Sanders et al.]
 - generalised stochastic Petri nets with confusion [Chiola et al.]
 - Markovian process algebras [Hermanns et al., Hillston et al.]

In this talk:

- ① Preliminary definitions.
- ② **Strong bisimulation** on CTMDPs.
- ③ Adaptation of **Continuous Stochastic Logic** (cf. CTL) to CTMDPs.
- ④ **Preservation** of CSL under strong bisimulation.

Definition (Continuous Time Markov Decision Process)

A CTMDP $(\mathcal{S}, \text{Act}, \mathbf{R}, \text{AP}, \text{L})$ has a finite set of states \mathcal{S} and propositions AP. $\text{L} : \mathcal{S} \rightarrow 2^{\text{AP}}$ is its state labelling. Further

- $\text{Act} = \{\alpha, \beta, \dots\}$ is a finite set of actions and
- $\mathbf{R} : \mathcal{S} \times \text{Act} \times \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}$ is a transition rate matrix such that
 - $\mathbf{R}(s, \alpha, s') = \lambda$ is the rate of a negative exponential distribution

$$f_X(t) = \begin{cases} \lambda \cdot e^{-\lambda \cdot t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad E[X] = \frac{1}{\lambda}$$

such that $\text{Act}(s) = \{\alpha \in \text{Act} \mid \exists s' \in \mathcal{S}. \mathbf{R}(s, \alpha, s') > 0\} \neq \emptyset$ for all $s \in \mathcal{S}$.

- $E(s, \alpha) = \sum_{s' \in \mathcal{S}} \mathbf{R}(s, \alpha, s')$ is the **exit rate** of s under α .

Example

Continuous action in Act(s) = $\mathbb{R}_{\geq 0}$

Continuous time between propositions

Continuous probability between states

Continuous time between actions

Continuous probability between propositions

Definition (Continuous Time Markov Decision Process)

A CTMDP $(\mathcal{S}, \text{Act}, \mathbf{R}, \text{AP}, \text{L})$ has a finite set of states \mathcal{S} and propositions AP. $\text{L} : \mathcal{S} \rightarrow 2^{\text{AP}}$ is its state labelling. Further

- $\text{Act} = \{\alpha, \beta, \dots\}$ is a finite set of actions and
- $\mathbf{R} : \mathcal{S} \times \text{Act} \times \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}$ is a transition rate matrix such that
 - $\mathbf{R}(s, \alpha, s') = \lambda$ is the rate of a negative exponential distribution

$$f_X(t) = \begin{cases} \lambda \cdot e^{-\lambda \cdot t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad E[X] = \frac{1}{\lambda}$$

such that $\text{Act}(s) = \{\alpha \in \text{Act} \mid \exists s' \in \mathcal{S}. \mathbf{R}(s, \alpha, s') > 0\} \neq \emptyset$ for all $s \in \mathcal{S}$.

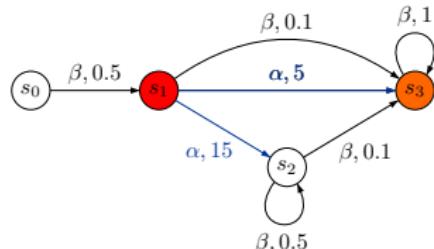
- $E(s, \alpha) = \sum_{s' \in \mathcal{S}} \mathbf{R}(s, \alpha, s')$ is the **exit rate** of s under α .

Example

- ① Choose action in $\text{Act}(s_1) = \{\alpha, \beta\}$
nondeterministically: say α

- ② **Race condition** between α -transitions:

- Mean delay: $\frac{1}{\mathbf{R}(s_1, \alpha, s_2) + \mathbf{R}(s_1, \alpha, s_3)} = \frac{1}{20}$
- Probability to move to s_3 : $\frac{\mathbf{R}(s_1, \alpha, s_3)}{E(s, \alpha)} = \frac{1}{4}$



Transitions and paths

Definition (Paths)

① **Finite paths** of length $n \in \mathbb{N}$ are denoted

$$\pi = s_0 \xrightarrow{\alpha_0, t_0} s_1 \xrightarrow{\alpha_1, t_1} s_2 \xrightarrow{\alpha_2, t_2} \dots \xrightarrow{\alpha_{n-1}, t_{n-1}} s_n$$

- Paths n is the set of paths of length n and
- $\pi \downarrow = s_n$ is the last state of π .

② Paths $^\omega$ is the corresponding set of **infinite paths**

- $\pi @ t$ is the state occupied at time t on path π .

Definition (Conditional transitions)

Given a state s and an action a , the set of transitions $\{s \xrightarrow{a} s'\}$ is called the **conditional transitions** of s for a .

Given a state s and an action a , the set of transitions $\{s \xrightarrow{a} s'\}$ is called the **conditional transitions** of s for a .

Given a state s and an action a , the set of transitions $\{s \xrightarrow{a} s'\}$ is called the **conditional transitions** of s for a .

Given a state s and an action a , the set of transitions $\{s \xrightarrow{a} s'\}$ is called the **conditional transitions** of s for a .

Given a state s and an action a , the set of transitions $\{s \xrightarrow{a} s'\}$ is called the **conditional transitions** of s for a .

Given a state s and an action a , the set of transitions $\{s \xrightarrow{a} s'\}$ is called the **conditional transitions** of s for a .

Given a state s and an action a , the set of transitions $\{s \xrightarrow{a} s'\}$ is called the **conditional transitions** of s for a .

Transitions and paths

Definition (Paths)

① **Finite paths** of length $n \in \mathbb{N}$ are denoted

$$\pi = s_0 \xrightarrow{\alpha_0, t_0} s_1 \xrightarrow{\alpha_1, t_1} s_2 \xrightarrow{\alpha_2, t_2} \dots \xrightarrow{\alpha_{n-1}, t_{n-1}} s_n$$

- Paths n is the set of paths of length n and
- $\pi \downarrow = s_n$ is the last state of π .

② Paths $^\omega$ is the corresponding set of **infinite paths**

- $\pi @ t$ is the state occupied at time t on path π .

Definition (Combined transition)

A **combined transition** $m = (\alpha, t, s')$

- α is the action (chosen externally),
- t is the transition's **firing time** and
- s' the transition's **successor state**.

$\Omega := \text{Act} \times \mathbb{R}_{\geq 0} \times \mathcal{S}$ is the set of combined transitions.



Definition (Measurable spaces)

Probability measures are defined on σ -fields

- ① \mathfrak{F} of sets of **combined transitions**:

$$\Omega := \text{Act} \times \mathbb{R}_{\geq 0} \times \mathcal{S}$$

$\mathfrak{B}(\mathbb{R}_{\geq 0})$: Borel σ -field for $\mathbb{R}_{\geq 0}$

$$\mathfrak{F} := \sigma(\mathfrak{F}_{\text{Act}} \times \mathfrak{B}(\mathbb{R}_{\geq 0}) \times \mathfrak{F}_{\mathcal{S}})$$

• $\mathfrak{F}_{\text{Act}}$ of sets of actions (deterministic, nondeterministic, discrete, continuous, ...)

• $\mathfrak{F}_{\mathcal{S}}$ of sets of states (deterministic, nondeterministic, discrete, continuous, ...)

• $\mathfrak{F}_{\mathbb{R}_{\geq 0}}$ of sets of time (deterministic, nondeterministic, discrete, continuous, ...)

• $\mathfrak{F}_{\text{combined}}$ of sets of combined transitions (deterministic, nondeterministic, discrete, continuous, ...)

The σ -field \mathfrak{F} is closed under

- union
- intersection
- complement

Definition (Measurable spaces)

Probability measures are defined on σ -fields

- ① \mathfrak{F} of sets of **combined transitions**:

$$\Omega := \text{Act} \times \mathbb{R}_{\geq 0} \times \mathcal{S}$$

$\mathfrak{B}(\mathbb{R}_{\geq 0})$: Borel σ -field for $\mathbb{R}_{\geq 0}$

$$\mathfrak{F} := \sigma(\mathfrak{F}_{\text{Act}} \times \mathfrak{B}(\mathbb{R}_{\geq 0}) \times \mathfrak{F}_{\mathcal{S}})$$

- ② $\mathfrak{F}_{\text{Paths}^n}$ of sets of **paths of length n** :

$$\mathfrak{F}_{\text{Paths}^n} := \sigma(\{S_0 \times M_1 \times \dots \times M_n \mid S_0 \in \mathfrak{F}_{\mathcal{S}}, M_i \in \mathfrak{F}\})$$

→ [View slides](#)

Definition (Measurable spaces)

Probability measures are defined on σ -fields

① \mathfrak{F} of sets of **combined transitions**:

$$\Omega := \text{Act} \times \mathbb{R}_{\geq 0} \times \mathcal{S}$$

$\mathfrak{B}(\mathbb{R}_{\geq 0})$: Borel σ -field for $\mathbb{R}_{\geq 0}$

$$\mathfrak{F} := \sigma(\mathfrak{F}_{\text{Act}} \times \mathfrak{B}(\mathbb{R}_{\geq 0}) \times \mathfrak{F}_{\mathcal{S}})$$

② $\mathfrak{F}_{\text{Paths}^n}$ of sets of **paths of length n** :

$$\mathfrak{F}_{\text{Paths}^n} := \sigma(\{S_0 \times M_1 \times \dots \times M_n \mid S_0 \in \mathfrak{F}_{\mathcal{S}}, M_i \in \mathfrak{F}\})$$

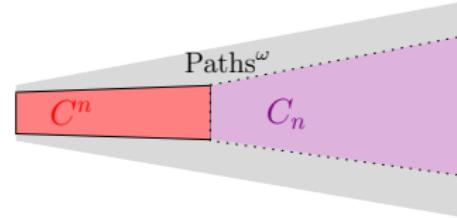
③ $\mathfrak{F}_{\text{Paths}^\omega}$ of sets of **infinite paths**:

Cylinder set construction:

- Any $C^n \in \mathfrak{F}_{\text{Paths}^n}$ defines a **cylinder base** (of finite length)
- $C_n := \{\pi \in \text{Paths}^\omega \mid \pi[0..n] \in C^n\}$ is a **cylinder** (extension to infinity).

The σ -field $\mathfrak{F}_{\text{Paths}^\omega}$ is then

$$\mathfrak{F}_{\text{Paths}^\omega} := \sigma\left(\bigcup_{n=0}^{\infty} \{C_n \mid C^n \in \mathfrak{F}_{\text{Paths}^n}\}\right)$$



Defining probability measures

Schedulers resolve nondeterministic choices between actions. Classes are

- ① either **deterministic** or **randomized** and
- ② depending on available **history**.

Definition of measurable scheduler

A measurable scheduler \mathcal{S} is a function

$$\mathcal{S}: \text{Paths} \times \mathcal{A} \rightarrow [0, 1] \quad \text{where}$$

• \mathcal{A} is the domain of actions for \mathcal{S}

• \mathcal{S} is measurable for \mathcal{A} & \mathcal{S}

Example (Why such schedulers?)

Stochastic schedulers are not optimal

→ 1. Let's choose 0 to maximize probability

→ 2. Let's choose 1 to minimize probability

Defining probability measures

Schedulers resolve nondeterministic choices between actions. Classes are

- ① either **deterministic** or **randomized** and
- ② depending on available **history**.

Definition (Measurable scheduler)

A measurable scheduler [WJ, 2006] is a mapping

$$\mathcal{D} : \text{Paths}^* \times \mathfrak{F}_{\text{Act}} \rightarrow [0, 1] \quad \text{where:}$$

- ① $\mathcal{D}(\pi, \cdot) \in \text{Distr}(\text{Act}(\pi \downarrow))$ for $\pi \in \text{Paths}^*$
- ② $\mathcal{D}(\cdot, A)$ are measurable for $A \in \mathfrak{F}_{\text{Act}}$.

Example (Why such schedulers?)

Stochastic schedulers are not optimal

→ choose \mathcal{D} to maximize probability

→ leads to suboptimal schedulers

Defining probability measures

Schedulers resolve nondeterministic choices between actions. Classes are

- ① either **deterministic** or **randomized** and
- ② depending on available **history**.

Definition (Measurable scheduler)

A measurable scheduler [WJ, 2006] is a mapping

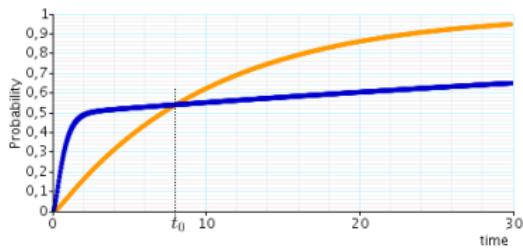
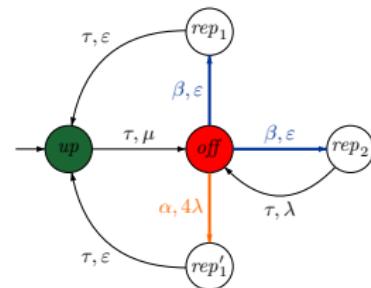
$$\mathcal{D} : \text{Paths}^* \times \mathfrak{F}_{\text{Act}} \rightarrow [0, 1] \quad \text{where:}$$

- ① $\mathcal{D}(\pi, \cdot) \in \text{Distr}(\text{Act}(\pi \downarrow))$ for $\pi \in \text{Paths}^*$
- ② $\mathcal{D}(\cdot, A)$ are measurable for $A \in \mathfrak{F}_{\text{Act}}$.

Example (Why such schedulers?)

Stationary schedulers are not optimal:

- $t \leq t_0$: choose β to maximize probability
- $t > t_0$: now start choosing α



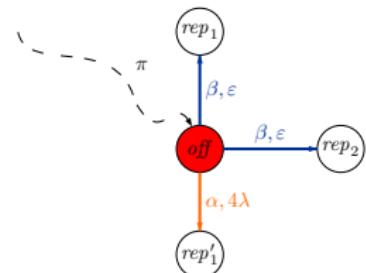
Example (One-step probabilities)

The event to go from off to rep_1 in 2 time units:

$$M = \{\beta\} \times [0, 2] \times \{rep_1\} \in \mathfrak{F}.$$

Its probability $\mu_{\mathcal{D}}(\pi, M)$ depends on:

- ① the probability $\mathcal{D}(\pi, \{\beta\})$ of choosing β



↳ The state 'off' is active in the first time unit.

↳ The state 'rep₁' is active in the second time unit.

Example (One-step probabilities)

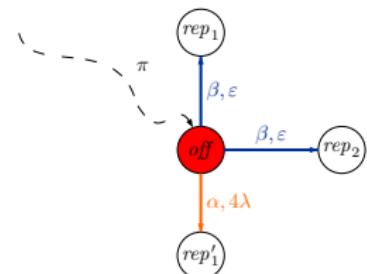
The event to go from off to rep_1 in 2 time units:

$$M = \{\beta\} \times [0, 2] \times \{rep_1\} \in \mathfrak{F}.$$

Its probability $\mu_{\mathcal{D}}(\pi, M)$ depends on:

- ① the probability $\mathcal{D}(\pi, \{\beta\})$ of choosing β
- ② the exp. distr. **residence time** in off :

$$\eta_{E(off, \beta)}(t) = E(off, \beta) \cdot e^{-E(off, \beta) \cdot t} = 2\varepsilon \cdot e^{-2\varepsilon \cdot t}$$



Example (One-step probabilities)

The event to go from off to rep_1 in 2 time units:

$$M = \{\beta\} \times [0, 2] \times \{rep_1\} \in \mathfrak{F}.$$

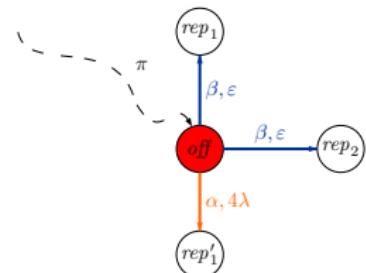
Its probability $\mu_{\mathcal{D}}(\pi, M)$ depends on:

- ① the probability $\mathcal{D}(\pi, \{\beta\})$ of choosing β
- ② the exp. distr. **residence time** in off :

$$\eta_{E(off, \beta)}(t) = E(off, \beta) \cdot e^{-E(off, \beta) \cdot t} = 2\varepsilon \cdot e^{-2\varepsilon \cdot t}$$

- ③ the **race** between rep_1 and rep_2 :

$$\mathbf{P}(off, \beta, rep_1) = \frac{\mathbf{R}(off, \beta, rep_1)}{E(off, \beta)} = \frac{\varepsilon}{2\varepsilon} = 0.5.$$



Example (One-step probabilities)

The event to go from off to rep_1 in 2 time units:

$$M = \{\beta\} \times [0, 2] \times \{rep_1\} \in \mathfrak{F}.$$

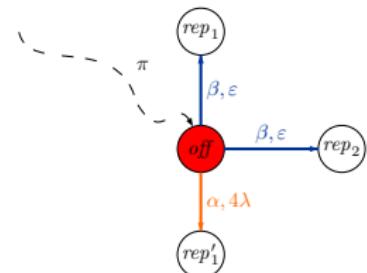
Its probability $\mu_{\mathcal{D}}(\pi, M)$ depends on:

- ① the probability $\mathcal{D}(\pi, \{\beta\})$ of choosing β
- ② the exp. distr. **residence time** in off :

$$\eta_{E(off, \beta)}(t) = E(off, \beta) \cdot e^{-E(off, \beta) \cdot t} = 2\varepsilon \cdot e^{-2\varepsilon \cdot t}$$

- ③ the **race** between rep_1 and rep_2 :

$$\mathbf{P}(off, \beta, rep_1) = \frac{\mathbf{R}(off, \beta, rep_1)}{E(off, \beta)} = \frac{\varepsilon}{2\varepsilon} = 0.5.$$



Definition (Measuring sets of combined transitions)

For history $\pi \in \text{Paths}^*$ and $\mathcal{D} \in \text{THR}$ define **probability measure** $\mu_{\mathcal{D}}(\pi, \cdot)$ on \mathfrak{F} :

$$\mu_{\mathcal{D}}(\pi, M) := \int_{\text{Act}} \mathcal{D}(\pi, d\alpha) \int_{\mathbb{R}_{\geq 0}} \eta_{E(\pi \downarrow, \alpha)}(dt) \int_{\mathcal{S}} \mathbf{I}_M(\alpha, t, s) \mathbf{P}(\pi \downarrow, \alpha, ds).$$

Definition (Measuring sets of paths)

① Initial distribution ν :

The probability to start in state s is $\nu(\{s\}) = \nu(s)$.

② Paths on sets of states

Definition (Bisimulation) and Definition (CTMDP) define inductively

$$\mathbb{P}_{\nu, \pi}(S) = \sum_{s \in S} \nu(s)$$

$$\mathbb{P}_{\nu, \pi}(S) = \left\{ \begin{array}{l} \mathbb{P}_{\nu, \pi}(A) \mid \text{ for all } a \in A \end{array} \right\}$$

③ Paths on sets of paths

Definition (Bisimulation) and Definition (CTMDP) define inductively

→ The probability of a bisimulation relation between sets of paths

$$\mathbb{P}_{\nu, \pi}(S) = \mathbb{P}_{\nu, \pi}(T)$$

The extended property

Definition (Measuring sets of paths)

① Initial distribution ν :

The probability to start in state s is $\nu(\{s\}) = \nu(s)$.

② $\Pr_{\nu, \mathcal{D}}^n$ on sets of finite paths:

Let $\nu \in \text{Distr}(\mathcal{S})$ and $\mathcal{D} \in \text{THR}$. Define inductively:

$$\Pr_{\nu, \mathcal{D}}^0(\Pi) := \sum_{s \in \Pi} \nu(s)$$

$$\Pr_{\nu, \mathcal{D}}^{n+1}(\Pi) := \int_{\text{Paths}^n} \Pr_{\nu, \mathcal{D}}^n(d\pi) \int_{\Omega} \mathbf{I}_{\Pi}(\pi \circ m) \mu_{\mathcal{D}}(\pi, dm)$$

Probability of sets of paths

Definition (Measuring sets of paths)

① Initial distribution ν :

The probability to start in state s is $\nu(\{s\}) = \nu(s)$.

② $\Pr_{\nu, \mathcal{D}}^n$ on sets of finite paths:

Let $\nu \in \text{Distr}(\mathcal{S})$ and $\mathcal{D} \in \text{THR}$. Define inductively:

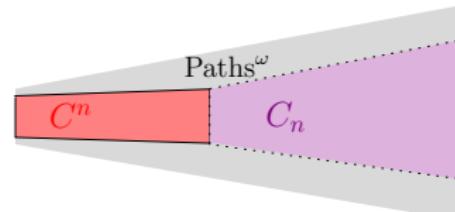
$$\Pr_{\nu, \mathcal{D}}^0(\Pi) := \sum_{s \in \Pi} \nu(s)$$

$$\Pr_{\nu, \mathcal{D}}^{n+1}(\Pi) := \int_{\text{Paths}^n} \Pr_{\nu, \mathcal{D}}^n(d\pi) \int_{\Omega} \mathbf{I}_{\Pi}(\pi \circ m) \mu_{\mathcal{D}}(\pi, dm)$$

③ $\Pr_{\nu, \mathcal{D}}^{\omega}$ on sets of infinite paths:

- A **cylinder base** is a measurable set $C^n \in \mathfrak{F}_{\text{Paths}^n}$
- C^n defines **cylinder** $C_n = \{\pi \in \text{Paths}^{\omega} \mid \pi[0..n] \in C^n\}$
- The probability of cylinder C_n is that of its base C^n :

$$\Pr_{\nu, \mathcal{D}}^{\omega}(C_n) = \Pr_{\nu, \mathcal{D}}^n(C^n).$$



This extends to $\mathfrak{F}_{\text{Paths}^{\omega}}$ by **Ionescu-Tulcea**.

Definition (Strong bisimulation relation)

Equivalence $\mathcal{R} \subseteq \mathcal{S} \times \mathcal{S}$ is a **strong bisimulation relation**

iff for all $(u, v) \in \mathcal{R}$ and all $\alpha \in \text{Act}$:

- ① $L(u) = L(v)$ and
- ② $\forall C \in \mathcal{S}_{\mathcal{R}}. \quad \mathbf{R}(u, \alpha, C) = \mathbf{R}(v, \alpha, C)$.

$$\mathbf{R}(u, \alpha, C) := \sum_{u' \in C} \mathbf{R}(u, \alpha, u')$$

u, v are **strongly bisimilar** ($u \sim v$) iff \exists str. bisim. relation \mathcal{R} with $(u, v) \in \mathcal{R}$.

Example: $\mathcal{S} = \{s_1, s_2, s_3\}$

For $\mathcal{A} = (A, \text{Act}, \text{Ran}, \text{P}, \text{P}^{\text{d}}, \text{P}^{\text{u}})$ with $A = \{a_1, a_2, a_3\}$ and $\text{Ran} = (S, A, \text{Ran}, \text{P}, \text{P}^{\text{d}}, \text{P}^{\text{u}})$

$$\text{Ran} = (S, A, \text{Ran}, \text{P}, \text{P}^{\text{d}}, \text{P}^{\text{u}})$$

$$S = \{s_1, s_2, s_3\}$$

$$\text{Ran} = (S, A, \text{Ran}, \text{P}, \text{P}^{\text{d}}, \text{P}^{\text{u}})$$

Example

Definition (Strong bisimulation relation)

Equivalence $\mathcal{R} \subseteq \mathcal{S} \times \mathcal{S}$ is a **strong bisimulation relation**

iff for all $(u, v) \in \mathcal{R}$ and all $\alpha \in \text{Act}$:

- ① $L(u) = L(v)$ and
- ② $\forall C \in \mathcal{S}_{\mathcal{R}}. \quad \mathbf{R}(u, \alpha, C) = \mathbf{R}(v, \alpha, C)$.

$$\mathbf{R}(u, \alpha, C) := \sum_{u' \in C} \mathbf{R}(u, \alpha, u')$$

u, v are **strongly bisimilar** ($u \sim v$) iff \exists str. bisim. relation \mathcal{R} with $(u, v) \in \mathcal{R}$.

Definition (Quotient under \sim)

For $\mathcal{C} = (\mathcal{S}, \text{Act}, \mathbf{R}, \text{AP}, \text{L})$, its **quotient under \sim** is $\tilde{\mathcal{C}} := (\tilde{\mathcal{S}}, \text{Act}, \tilde{\mathbf{R}}, \text{AP}, \tilde{\text{L}})$:

$$\tilde{\text{L}}([s]) := \text{L}(s)$$

$$\tilde{\mathcal{S}} := \{[s]_{\sim} \mid s \in \mathcal{S}\}$$

$$\tilde{\mathbf{R}}([s], \alpha, C) := \mathbf{R}(s, \alpha, C)$$

Example

Definition (Strong bisimulation relation)

Equivalence $\mathcal{R} \subseteq \mathcal{S} \times \mathcal{S}$ is a **strong bisimulation relation**

iff for all $(u, v) \in \mathcal{R}$ and all $\alpha \in \text{Act}$:

- ① $L(u) = L(v)$ and
- ② $\forall C \in \mathcal{S}_{\mathcal{R}}. \quad \mathbf{R}(u, \alpha, C) = \mathbf{R}(v, \alpha, C)$.

$$\mathbf{R}(u, \alpha, C) := \sum_{u' \in C} \mathbf{R}(u, \alpha, u')$$

u, v are **strongly bisimilar** ($u \sim v$) iff \exists str. bisim. relation \mathcal{R} with $(u, v) \in \mathcal{R}$.

Definition (Quotient under \sim)

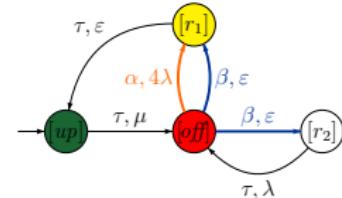
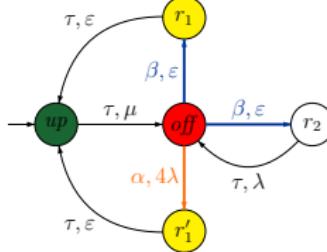
For $\mathcal{C} = (\mathcal{S}, \text{Act}, \mathbf{R}, \text{AP}, \text{L})$, its **quotient under \sim** is $\tilde{\mathcal{C}} := (\tilde{\mathcal{S}}, \text{Act}, \tilde{\mathbf{R}}, \text{AP}, \tilde{\text{L}})$:

$$\tilde{\text{L}}([s]) := \text{L}(s)$$

$$\tilde{\mathcal{S}} := \{[s]_{\sim} \mid s \in \mathcal{S}\}$$

$$\tilde{\mathbf{R}}([s], \alpha, C) := \mathbf{R}(s, \alpha, C)$$

Example



Continuous Stochastic Logic

[Aziz et al. 2000, Baier et al. 2003]

Example (Transient state formula)

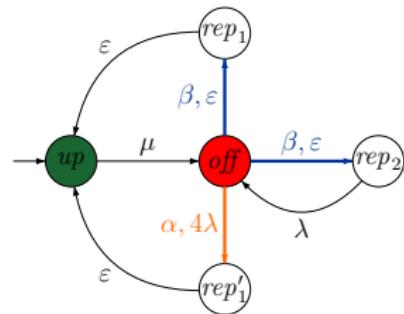
In state *off*, the probability to reach the *up* state within 20 time units **exceeds 0.5** under **any scheduling decision**:

$$\Phi = \text{off} \rightarrow \forall^{>0.5} \Diamond^{[0,20]} \text{up}$$

Example (Long-run average [de Alfaro, LICS 98])

For any scheduler, the system on average is not operational for **less than 1%** of its execution time:

$$\Psi = \text{L}^{<0.01} \neg \text{up}$$



Definition (Syntax of CSL)

For $a \in AP$, $p \in [0, 1]$, $I \subseteq \mathbb{R}_{\geq 0}$ a nonempty interval and $\sqsubseteq \in \{<, \leq, \geq, >\}$, **state formulas** and **path formulas** are:

$$\Phi ::= a \mid \neg\Phi \mid \Phi \wedge \Phi \mid \forall^{\sqsubseteq p} \varphi \mid \mathsf{L}^{\sqsubseteq p} \Phi \quad \varphi ::= X^I \Phi \mid \Phi \mathsf{U}^I \Phi.$$

Definition (Semantics of path formulas)

For CTMDP \mathcal{G} and infinite path $\pi = \langle \pi^0, \pi^1, \pi^2, \dots \rangle$ the semantics is given by

$$\pi \models \forall^{\leq p} \varphi \quad \text{iff} \quad \forall i \in \mathbb{N} \quad \varphi(\pi^i) \in \mathcal{G}$$

$$\pi \models \mathsf{L}^{\leq p} \varphi \quad \text{iff} \quad \exists i \in \mathbb{N} \quad \text{such that} \quad \varphi(\pi^i) \in \mathcal{G} \quad \text{and} \quad \forall j \in \mathbb{N} \quad i \leq j \leq p$$

Example

Let $\mathcal{G} = \text{CTMDP}$ and φ be Path formulas as follows

Definition (Syntax of CSL)

For $a \in AP$, $p \in [0, 1]$, $I \subseteq \mathbb{R}_{\geq 0}$ a nonempty interval and $\sqsubseteq \in \{<, \leq, \geq, >\}$, **state formulas** and **path formulas** are:

$$\Phi ::= a \mid \neg\Phi \mid \Phi \wedge \Phi \mid \forall^{\sqsubseteq p} \varphi \mid \mathbf{L}^{\sqsubseteq p} \Phi \quad \varphi ::= X^I \Phi \mid \Phi U^I \Phi.$$

Definition (Semantics of path formulas)

For CTMDP \mathcal{C} and infinite path $\pi = s_0 \xrightarrow{\alpha_0, t_0} s_1 \xrightarrow{\alpha_1, t_1} \dots$ define:

$$\pi \models X^I \Phi \iff \pi[1] \models \Phi \wedge t_0 \in I$$

$$\pi \models \Phi U^I \Psi \iff \exists t \in I. (\pi @ t \models \Psi \wedge (\forall t' \in [0, t). \pi @ t' \models \Phi)).$$

Example

Let $\pi = \text{idle} \circ \text{idle} \circ \text{idle} \circ \dots$ and $\varphi \in \text{Path formulas}$ as follows:

Definition (Syntax of CSL)

For $a \in AP$, $p \in [0, 1]$, $I \subseteq \mathbb{R}_{\geq 0}$ a nonempty interval and $\sqsubseteq \in \{<, \leq, \geq, >\}$, **state formulas** and **path formulas** are:

$$\Phi ::= a \mid \neg\Phi \mid \Phi \wedge \Phi \mid \forall^{\sqsubseteq p} \varphi \mid \mathsf{L}^{\sqsubseteq p} \Phi \quad \varphi ::= X^I \Phi \mid \Phi U^I \Phi.$$

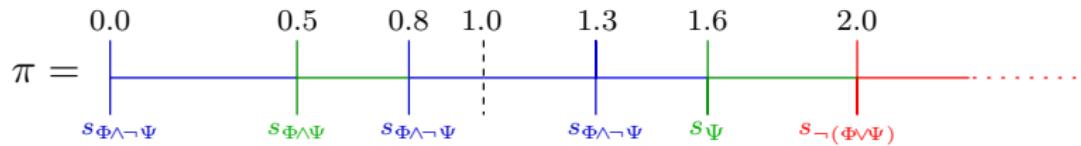
Definition (Semantics of path formulas)

For CTMDP \mathcal{C} and infinite path $\pi = s_0 \xrightarrow{\alpha_0, t_0} s_1 \xrightarrow{\alpha_1, t_1} \dots$ define:

$$\begin{aligned} \pi \models X^I \Phi &\iff \pi[1] \models \Phi \wedge t_0 \in I \\ \pi \models \Phi U^I \Psi &\iff \exists t \in I. (\pi @ t \models \Psi \wedge (\forall t' \in [0, t). \pi @ t' \models \Phi)). \end{aligned}$$

Example

Let $\varphi = \Phi U^{[1,2]} \Psi$ and $\pi \in \text{Paths}^\omega$ as follows:



Average residence time

For state formula Φ , path $\pi = s_0 \xrightarrow{\alpha_0, t_0} s_1 \xrightarrow{\alpha_1, t_1} s_2 \dots$ and time point t :

$$\text{avg}(\Phi, t, \pi) := \frac{1}{t} \int_0^t \mathbf{I}_{\text{Sat}(\Phi)}(\pi @ t') dt'.$$

$$\mathbf{I}_{\text{Sat}(\Phi)}(s) := \begin{cases} 1 & \text{if } s \in \text{Sat}(\Phi) \\ 0 & \text{otherwise} \end{cases}$$

Example

The average time spent in $\text{Sat}(\Phi)$ -states up to time $t = 3$:

$$\text{avg}(\Phi, 3, \pi) = \frac{1}{3} \int_0^3 \mathbf{I}_{\text{Sat}(\Phi)}(\pi @ t') dt'$$

Definition (Semantics of state formulas)

Let $\Phi \in \{0, 1\}$, $\Sigma = \{a, b, c, d\}$ and Φ state and path formulas:

$$\begin{aligned} \vdash \Phi_1 \rightarrow \Phi_2 &\iff \forall \pi \in \text{Paths} \exists t \in \mathbb{R} \forall t' \in [0, t] \Phi_1 @ t' \rightarrow \Phi_2 @ t' \\ \vdash \Phi_1 \wedge \Phi_2 &\iff \forall \pi \in \text{Paths} \forall t \in \mathbb{R} \Phi_1 @ t \wedge \Phi_2 @ t \end{aligned}$$

Average residence time

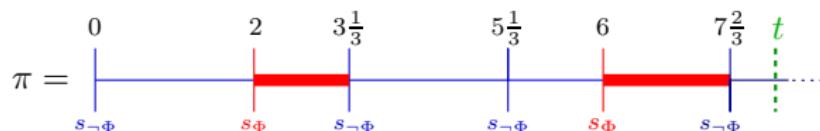
For state formula Φ , path $\pi = s_0 \xrightarrow{\alpha_0, t_0} s_1 \xrightarrow{\alpha_1, t_1} s_2 \dots$ and time point t :

$$\text{avg}(\Phi, t, \pi) := \frac{1}{t} \int_0^t \mathbf{I}_{\text{Sat}(\Phi)}(\pi @ t') dt'.$$

$\mathbf{I}_{\text{Sat}(\Phi)}(s) := \begin{cases} 1 & \text{if } s \in \text{Sat}(\Phi) \\ 0 & \text{otherwise} \end{cases}$

Example

The average time spent in $\text{Sat}(\Phi)$ -states up to time $t = 8$:



$$\text{avg}(\Phi, 8, \pi) = \frac{\frac{4}{3} + \frac{5}{3}}{8} = \frac{3}{8}$$

Average residence time

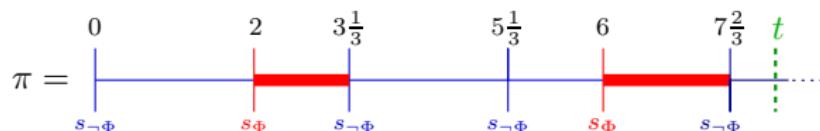
For state formula Φ , path $\pi = s_0 \xrightarrow{\alpha_0, t_0} s_1 \xrightarrow{\alpha_1, t_1} s_2 \dots$ and time point t :

$$\text{avg}(\Phi, t, \pi) := \frac{1}{t} \int_0^t \mathbf{I}_{\text{Sat}(\Phi)}(\pi @ t') dt'.$$

$\mathbf{I}_{\text{Sat}(\Phi)}(s) := \begin{cases} 1 & \text{if } s \in \text{Sat}(\Phi) \\ 0 & \text{otherwise} \end{cases}$

Example

The average time spent in $\text{Sat}(\Phi)$ -states up to time $t = 8$:



$$\text{avg}(\Phi, 8, \pi) = \frac{\frac{4}{3} + \frac{5}{3}}{8} = \frac{3}{8}$$

Definition (Semantics of state formulas)

Let $p \in [0, 1]$, $\sqsubseteq \in \{\leq, <, \geq, >\}$ and Φ, φ state and path formulas:

$$s \models \mathsf{L}^{\sqsubseteq p} \Phi \iff \forall D \in \text{THR.} \lim_{t \rightarrow \infty} \int_{\text{Paths}^\omega} \text{avg}(\Phi, t, \pi) \Pr_{\nu_s, D}^{\omega}(d\pi) \sqsubseteq p$$

$$s \models \forall^{\sqsubseteq p} \varphi \iff \forall D \in \text{THR.} \Pr_{\nu_s, D}^{\omega} \{ \pi \in \text{Paths}^\omega \mid \pi \models \varphi \} \sqsubseteq p$$

Strong bisimilarity preserves CSL properties

Theorem

Let $(\mathcal{S}, \text{Act}, \mathbf{R}, \text{AP}, \text{L})$ be a CTMDP. For all states $u, v \in \mathcal{S}$ it holds:

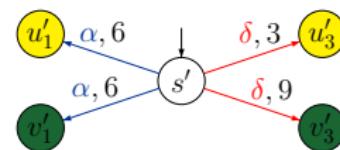
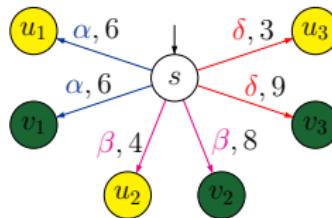
$$u \sim v \implies \forall \Phi \in \text{CSL}. (u \models \Phi \iff v \models \Phi).$$

Proof by structural induction:

- ① a, \neg, \wedge omitted
- ② $\forall^{\sqsubseteq p} \varphi$ sketch in this talk
- ③ $\text{L}^{\sqsubseteq p} \varphi$ straightforward, relies on $\forall^{\sqsubseteq p} \varphi$

The reverse conjecture does not hold:

Counterexample [Segala et al., Nordic J. of Comp. 1995, Baier 1998]



For all π it holds $\pi \vdash S \rightarrow S'$

For all π it holds $\pi \vdash S' \rightarrow S$

CSL formula ψ

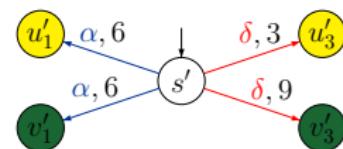
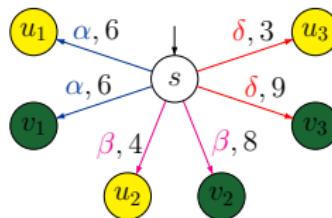
The choice of ψ is such that ψ contains an infinite supremum operator \sup

$\psi = \sup_{n \in \mathbb{N}} \psi_n$ where ψ_n is

$\psi_n = \exists \pi \forall \pi' \exists \pi'' \forall \pi''' \exists \pi'''' \forall \pi''''' \exists \pi'''''' \forall \pi'''''$

The reverse conjecture does not hold:

Counterexample [Segala et al., Nordic J. of Comp. 1995, Baier 1998]



① $s \not\sim s'$ as $\mathbf{R}(s, \beta, [u_2]) = 4$ whereas $\mathbf{R}(s', \beta, \cdot) = 0$.

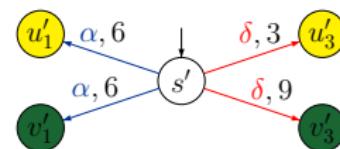
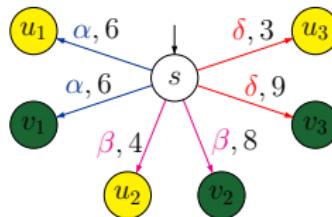
• The conjecture is false

• The choice of the state s does not contribute an infinite supremum

• The choice of the state s' does not contribute a minimum infimum

The reverse conjecture does not hold:

Counterexample [Segala et al., Nordic J. of Comp. 1995, Baier 1998]



- 1 $s \not\sim s'$ as $\mathbf{R}(s, \beta, [u_2]) = 4$ whereas $\mathbf{R}(s', \beta, \cdot) = 0$.
- 2 For all Φ it holds: $s \models \Phi \iff s' \models \Phi$:

- CSL can only express **extreme probability bounds**.
- The choice of **action β** does not contribute an infimum or supremum.
- Some examples:

- $\Phi = \exists^{\leq \frac{1}{3}} X^{[0, \infty)} \text{ yellow: choose } \delta$
- $\Phi = \exists^{\geq \frac{1}{3}} X^{[0, \infty)} \text{ yellow: choose } \alpha$

$$\exists^{\sqsubseteq p} \varphi \equiv \neg \forall^{\exists p} \varphi$$

Preservation of CSL formulas

Proof idea for transient state formulas

- Assume $\varphi \rightarrow \psi$ and $\psi \models_{\text{CTMDP}} \varphi$. To prove $\varphi \models_{\text{CTMDP}} \psi$, we

$$\text{Use induction on the rank of } \varphi \text{ (rank } \varphi = \text{rank } \psi \text{)} \quad (1)$$

- Let $\varphi \models_{\text{CTMDP}} \psi$ be a scheduler property.

- From 1. construct ψ (recursively) such that

$$\text{rank } \psi = \text{rank } \varphi \text{ and } \psi \models_{\text{CTMDP}} \varphi \text{ and } \psi \models_{\text{CTMDP}} \varphi \rightarrow \psi \quad (2)$$

- Since $\psi \models_{\text{CTMDP}} \varphi$ we obtain $\varphi \models_{\text{CTMDP}} \psi \rightarrow \varphi \models_{\text{CTMDP}} \psi$.

How to prove equation (2)

For a predicate φ of a set of paths

Preservation of CSL formulas

Proof idea for transient state formulas

- Assume $u \sim v$ and $u \models \forall \sqsubseteq^p \varphi$. To prove: $v \models \forall \sqsubseteq^p \varphi$, i.e.

$$\forall \mathcal{V} \in \text{THR}. \Pr_{\nu_v, \nu}^{\omega} \{ \pi \in \text{Paths}^{\omega} \mid \pi \models \varphi \} \sqsubseteq p. \quad (1)$$

- Let \mathcal{V} be the THR instance in (1).
- From \mathcal{V} construct \mathcal{V}' (copying) such that
 - \mathcal{V}' has the same set of states as \mathcal{V} and the same set of transitions.
 - States in \mathcal{V}' are partitioned into two sets: \mathcal{V}'_1 and \mathcal{V}'_2 .
 - States in \mathcal{V}'_1 are called u -states and states in \mathcal{V}'_2 are called v -states.
 - Transitions in \mathcal{V}' are copied from \mathcal{V} and are labeled with the same labels as in \mathcal{V} .
 - For each state $s \in \mathcal{V}'_1$ and each label a , if there is a transition $s \xrightarrow{a} t$ in \mathcal{V} , then there is a transition $s \xrightarrow{a} t$ in \mathcal{V}' with $t \in \mathcal{V}'_1$.
 - For each state $s \in \mathcal{V}'_2$ and each label a , if there is a transition $s \xrightarrow{a} t$ in \mathcal{V} , then there is a transition $s \xrightarrow{a} t$ in \mathcal{V}' with $t \in \mathcal{V}'_2$.
- States in \mathcal{V}'_1 are called u -states and states in \mathcal{V}'_2 are called v -states.

How to prove equation (2)

For a state s in \mathcal{V}'_1 and a set of paths

Preservation of CSL formulas

Proof idea for transient state formulas

- Assume $u \sim v$ and $u \models \forall \sqsubseteq^p \varphi$. To prove: $v \models \forall \sqsubseteq^p \varphi$, i.e.

$$\forall \mathcal{V} \in \text{THR}. \Pr_{\nu_v, \mathcal{V}}^{\omega} \{ \pi \in \text{Paths}^{\omega} \mid \pi \models \varphi \} \sqsubseteq p. \quad (1)$$

- Let $\mathcal{V} \in \text{THR}$ be a scheduler (w.r.t. ν_v).
- From \mathcal{V} , construct \mathcal{U} (w.r.t. ν_u) such that

$$\Pr_{\nu_u, \mathcal{U}}^{\omega} \{ \pi \in \text{Paths}^{\omega} \mid \pi \models \varphi \} = \Pr_{\nu_v, \mathcal{V}}^{\omega} \{ \pi \in \text{Paths}^{\omega} \mid \pi \models \varphi \}. \quad (2)$$

How to prove equation (2)

For a scheduler \mathcal{U} and a set of paths

Preservation of CSL formulas

Proof idea for transient state formulas

- Assume $u \sim v$ and $u \models \forall \sqsubseteq^p \varphi$. To prove: $v \models \forall \sqsubseteq^p \varphi$, i.e.

$$\forall \mathcal{V} \in \text{THR}. \Pr_{\nu_v, \mathcal{V}}^{\omega} \{ \pi \in \text{Paths}^{\omega} \mid \pi \models \varphi \} \sqsubseteq p. \quad (1)$$

- Let $\mathcal{V} \in \text{THR}$ be a scheduler (w.r.t. ν_v).
- From \mathcal{V} , construct \mathcal{U} (w.r.t. ν_u) such that

$$\Pr_{\nu_u, \mathcal{U}}^{\omega} \{ \pi \in \text{Paths}^{\omega} \mid \pi \models \varphi \} = \Pr_{\nu_v, \mathcal{V}}^{\omega} \{ \pi \in \text{Paths}^{\omega} \mid \pi \models \varphi \}. \quad (2)$$

- Since $u \models \forall \sqsubseteq^p \varphi$, we obtain $\Pr_{\nu_v, \mathcal{V}}^{\omega} \{ \pi \in \text{Paths}^{\omega} \mid \pi \models \varphi \} \sqsubseteq p$.

How to prove equation (2)

For a scheduler \mathcal{V} , simulation of sets of paths

Preservation of CSL formulas

Proof idea for transient state formulas

- Assume $u \sim v$ and $u \models \forall \sqsubseteq^p \varphi$. To prove: $v \models \forall \sqsubseteq^p \varphi$, i.e.

$$\forall \mathcal{V} \in \text{THR}. \Pr_{\nu_v, \mathcal{V}}^{\omega} \{ \pi \in \text{Paths}^{\omega} \mid \pi \models \varphi \} \sqsubseteq p. \quad (1)$$

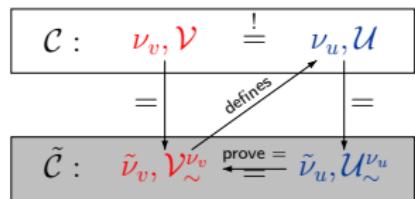
- Let $\mathcal{V} \in \text{THR}$ be a scheduler (w.r.t. ν_v).
- From \mathcal{V} , construct \mathcal{U} (w.r.t. ν_u) such that

$$\Pr_{\nu_u, \mathcal{U}}^{\omega} \{ \pi \in \text{Paths}^{\omega} \mid \pi \models \varphi \} = \Pr_{\nu_v, \mathcal{V}}^{\omega} \{ \pi \in \text{Paths}^{\omega} \mid \pi \models \varphi \}. \quad (2)$$

- Since $u \models \forall \sqsubseteq^p \varphi$, we obtain $\Pr_{\nu_v, \mathcal{V}}^{\omega} \{ \pi \in \text{Paths}^{\omega} \mid \pi \models \varphi \} \sqsubseteq p$.

How to prove equation (2)

For a specific subclass of sets of paths:
lift the argument to the quotient space



Preliminaries

Definition (Simple bisimulation closed sets of paths)

A measurable set of paths of the form

$$\Pi = [s_0] \times A_0 \times T_0 \times [s_1] \times \cdots \times A_{n-1} \times T_{n-1} \times [s_n]$$

is called **simple bisimulation closed**. Π corresponds to the set $\tilde{\Pi}$ on $\tilde{\mathcal{C}}$:

$$\tilde{\Pi} = \{[s_0]\} \times A_0 \times T_0 \times \{[s_1]\} \times \cdots \times A_{n-1} \times T_{n-1} \times \{[s_n]\}$$

Preliminary results

For initial distribution $\pi \in \mathcal{C}^{\text{initial}}$ and scheduler $\mathcal{D} \in \mathcal{C}^{\text{scheduler}}$

For initial distribution $\pi \in \mathcal{C}^{\text{initial}}$ and scheduler $\mathcal{D} \in \mathcal{C}^{\text{scheduler}}$

$$\mathbb{P}_{\mathcal{D}, \pi}(\mathbb{D}) = \mathbb{P}_{\mathcal{D}, \pi}(\mathbb{D})$$



Set of paths Π

Preliminaries

Definition (Simple bisimulation closed sets of paths)

A measurable set of paths of the form

$$\Pi = [s_0] \times A_0 \times T_0 \times [s_1] \times \cdots \times A_{n-1} \times T_{n-1} \times [s_n]$$

is called **simple bisimulation closed**. Π corresponds to the set $\tilde{\Pi}$ on $\tilde{\mathcal{C}}$:

$$\tilde{\Pi} = \{[s_0]\} \times A_0 \times T_0 \times \{[s_1]\} \times \cdots \times A_{n-1} \times T_{n-1} \times \{[s_n]\}$$

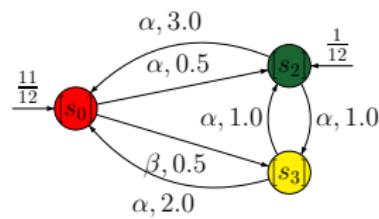
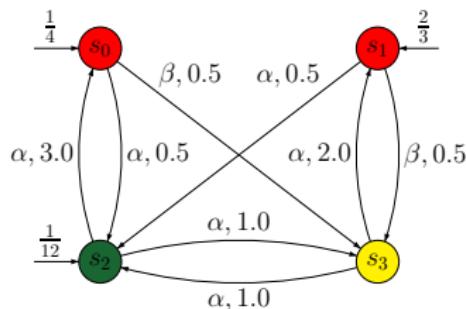
Preliminary goal

For initial distribution $\nu \in \text{Distr}(\mathcal{S})$ and scheduler $\mathcal{D} \in \text{THR}$:
 Provide a **measure preserving** scheduler \mathcal{D}_{\sim}^{ν} on $\tilde{\mathcal{C}}$ such that

$$\Pr_{\nu, \mathcal{D}}^{\omega}(\Pi) = \Pr_{\tilde{\nu}, \mathcal{D}_{\sim}^{\nu}}^{\omega}(\tilde{\Pi})$$

for **simple bisimulation closed** sets of paths Π .

Example



Let the Quotient scheduler be as follows:

Initially, the quotient scheduler starts in state s_0 and has the following transitions:

$s_0 \xrightarrow{\frac{11}{12}} s_0$ (self-loop)

$s_0 \xrightarrow{\alpha, 3.0} s_2$

$s_0 \xrightarrow{\alpha, 0.5} s_3$

$s_0 \xrightarrow{\beta, 0.5} s_3$

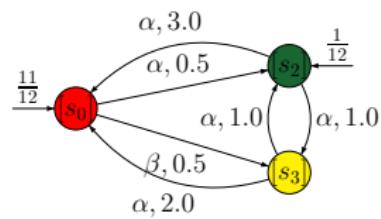
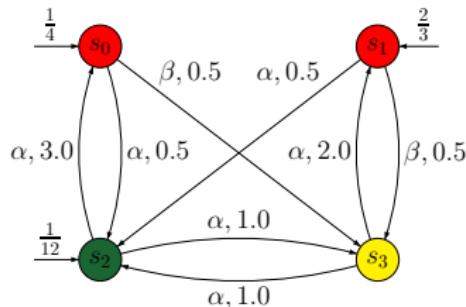
$s_2 \xrightarrow{\alpha, 1.0} s_0$

$s_2 \xrightarrow{\alpha, 1.0} s_3$

$s_3 \xrightarrow{\alpha, 2.0} s_0$

$s_3 \xrightarrow{\alpha, 1.0} s_2$

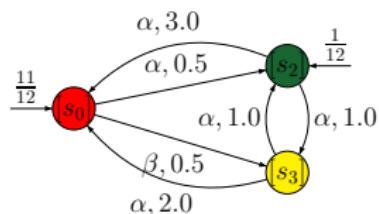
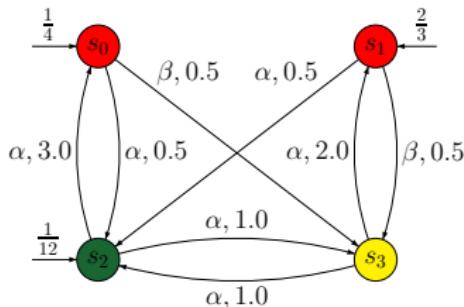
Example



Let the first decision of \mathcal{D} be as follows:

$$\mathcal{D}(s_0, \{\alpha\}) = \frac{2}{3} \quad \mathcal{D}(s_0, \{\beta\}) = \frac{1}{3} \quad \mathcal{D}(s_1, \{\alpha\}) = \frac{1}{4} \quad \mathcal{D}(s_1, \{\beta\}) = \frac{3}{4}$$

Example



Let the first decision of \mathcal{D} be as follows:

$$\mathcal{D}(s_0, \{\alpha\}) = \frac{2}{3} \quad \mathcal{D}(s_0, \{\beta\}) = \frac{1}{3} \quad \mathcal{D}(s_1, \{\alpha\}) = \frac{1}{4} \quad \mathcal{D}(s_1, \{\beta\}) = \frac{3}{4}$$

Intuitively, the quotient scheduler \mathcal{D}_\sim^ν then decides in $[s_0]$ as follows:

$$\mathcal{D}_\sim^\nu([s_0], \{\alpha\}) = \frac{\sum_{s \in [s_0]} \nu(s) \cdot \mathcal{D}(s, \{\alpha\})}{\sum_{s \in [s_0]} \nu(s)} = \frac{\frac{1}{4} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{4}}{\frac{1}{4} + \frac{2}{3}} = \frac{4}{11}$$

$$\mathcal{D}_\sim^\nu([s_0], \{\beta\}) = \frac{\sum_{s \in [s_0]} \nu(s) \cdot \mathcal{D}(s, \{\beta\})}{\sum_{s \in [s_0]} \nu(s)} = \frac{\frac{1}{4} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{3}{4}}{\frac{1}{4} + \frac{2}{3}} = \frac{7}{11}.$$

Quotient scheduler

Definition (History weight)

Given CTMDP \mathcal{C} , $\nu \in \text{Distr}(\mathcal{S})$ and $\mathcal{D} \in \text{THR}$.

Define the weight of history $\pi \in \text{Paths}^*$ inductively:

$$hw_0(\nu, \mathcal{D}, \pi) := \nu(\pi) \quad \text{if } \pi \in \text{Paths}^0 = \mathcal{S} \text{ and}$$

$$hw_{n+1}(\nu, \mathcal{D}, \pi \xrightarrow{\alpha_n, t_n} s_{n+1}) := hw_n(\nu, \mathcal{D}, \pi) \cdot \mathcal{D}(\pi, \{\alpha_n\}) \cdot \mathbf{P}(\pi \downarrow, \alpha_n, s_{n+1}).$$

Definition (Quotient scheduler)

For any history $\pi = (\pi_0, \pi_1, \dots, \pi_n)$ in Paths^* and the quotient scheduler \mathcal{D} , is defined by

$$\mathcal{D}(\pi, \{\alpha_n\}) = \bigcap_{\pi' \in \text{Paths}^*} \{ \pi' \mid \pi' \text{ is a history and } \pi' \downarrow = \pi \text{ and } \pi' \models \alpha_n \}$$

$$\text{where } \pi' = (\pi_0, \pi_1, \dots, \pi_n, \dots, \pi_{n-1}, \pi_n) \in \{ \pi_0 \} \times \{ \pi_1 \} \times \dots \times \{ \pi_n \} \times \{ \pi_n \}$$

Quotient scheduler

Definition (History weight)

Given CTMDP \mathcal{C} , $\nu \in \text{Distr}(\mathcal{S})$ and $\mathcal{D} \in \text{THR}$.

Define the weight of history $\pi \in \text{Paths}^*$ inductively:

$$hw_0(\nu, \mathcal{D}, \pi) := \nu(\pi) \quad \text{if } \pi \in \text{Paths}^0 = \mathcal{S} \text{ and}$$

$$hw_{n+1}(\nu, \mathcal{D}, \pi \xrightarrow{\alpha_n, t_n} s_{n+1}) := hw_n(\nu, \mathcal{D}, \pi) \cdot \mathcal{D}(\pi, \{\alpha_n\}) \cdot \mathbf{P}(\pi \downarrow, \alpha_n, s_{n+1}).$$

Definition (Quotient scheduler)

For any history $\tilde{\pi} = [s_0] \xrightarrow{\alpha_0, t_0} [s_1] \xrightarrow{\alpha_1, t_1} \dots \xrightarrow{\alpha_{n-1}, t_{n-1}} [s_n]$, the quotient scheduler \mathcal{D}_\sim^ν is defined by:

$$\mathcal{D}_\sim^\nu(\tilde{\pi}, \{\alpha_n\}) := \frac{\sum_{\pi \in \Pi} hw_n(\nu, \mathcal{D}, \pi) \cdot \mathcal{D}(\pi, \{\alpha_n\})}{\sum_{\pi \in \Pi} hw_n(\nu, \mathcal{D}, \pi)}$$

where $\Pi = [s_0] \times \{\alpha_0\} \times \{t_0\} \times \dots \times \{\alpha_{n-1}\} \times \{t_{n-1}\} \times [s_n]$.

Proof sketch I

Lemma

① For **simple bisimulation closed** sets of paths Π :

$$\Pr_{\nu, \mathcal{D}}^{\omega}(\Pi) = \Pr_{\tilde{\nu}, \mathcal{D}_{\sim}^{\nu}}^{\omega}(\tilde{\Pi}).$$

② For any path formula φ there exists a family $\{\Pi_k\}_{k \in \mathbb{N}}$ such that

$$\left\{ \pi \in \text{Paths}^{\omega} \mid \pi \models \varphi \right\} = \biguplus_{k=0}^{\infty} \Pi_k$$

and Π_k is simple bisimulation closed.

What have we gained?

For any $\Pi \in \text{Paths}^{\omega}$ $\Pr_{\nu, \mathcal{D}}^{\omega}(\Pi) = \Pr_{\tilde{\nu}, \mathcal{D}_{\sim}^{\nu}}^{\omega}(\tilde{\Pi})$

$\Pr_{\nu, \mathcal{D}}^{\omega}(\Pi) = \Pr_{\tilde{\nu}, \mathcal{D}_{\sim}^{\nu}}^{\omega}(\tilde{\Pi})$

Proof sketch I

Lemma

① For **simple bisimulation closed** sets of paths Π :

$$\Pr_{\nu, \mathcal{D}}^{\omega}(\Pi) = \Pr_{\tilde{\nu}, \mathcal{D}_{\sim}^{\nu}}^{\omega}(\tilde{\Pi}).$$

② For any path formula φ there exists a family $\{\Pi_k\}_{k \in \mathbb{N}}$ such that

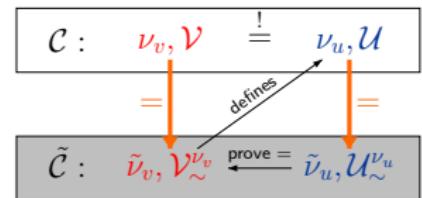
$$\left\{ \pi \in \text{Paths}^{\omega} \mid \pi \models \varphi \right\} = \biguplus_{k=0}^{\infty} \Pi_k$$

and Π_k is simple bisimulation closed.

What have we gained?

For any $\Pi_k \subseteq \{\pi \in \text{Paths}^{\omega} \mid \pi \models \varphi\}$:

$$\Pr_{\nu, \mathcal{D}}^{\omega}(\Pi_k) = \Pr_{\tilde{\nu}, \mathcal{D}_{\sim}^{\nu}}^{\omega}(\tilde{\Pi}_k).$$



Proof sketch II

Sketch of the proof

Let $\Pi = \{\pi \in \text{Paths}^\omega \mid \pi \models \varphi\}$. To show: $\text{Pr}_{\nu_u, \mathcal{U}}^\omega(\Pi) = \text{Pr}_{\nu_v, \mathcal{V}}^\omega(\Pi)$:

- ① Define scheduler \mathcal{U} to mimic $\mathcal{V}_{\sim}^{\nu_v}$ on the quotient:

$$\mathcal{U}\left(s_0 \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} s_n\right) := \mathcal{V}_{\sim}^{\nu_v}\left([s_0] \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} [s_n]\right).$$

Then $\mathcal{U} \sim \mathcal{V}_{\sim}^{\nu_v}$ and $\mathcal{U} \models \varphi$.

Proof sketch:
1. $\mathcal{U} \sim \mathcal{V}_{\sim}^{\nu_v}$
2. $\mathcal{U} \models \varphi$

Now we obtain the proof.

Proof sketch II

Sketch of the proof

Let $\Pi = \{\pi \in \text{Paths}^\omega \mid \pi \models \varphi\}$. To show: $\Pr_{\nu_u, \mathcal{U}}^\omega(\Pi) = \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi)$:

- ① Define scheduler \mathcal{U} to mimic $\mathcal{V}_{\sim}^{\nu_v}$ on the quotient:

$$\mathcal{U}\left(s_0 \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} s_n\right) := \mathcal{V}_{\sim}^{\nu_v}\left([s_0] \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} [s_n]\right).$$

- ② Then $\mathcal{U}_{\sim}^{\nu_u} = \mathcal{V}_{\sim}^{\nu_v}$ and $\tilde{\nu}_u = \tilde{\nu}_v$:

$$\mathcal{U}_{\sim}^{\nu_u}(\tilde{\pi}, \alpha_n) = \frac{\sum_{\pi \in \Pi} h_{w_n}(\nu_u, \mathcal{U}, \pi) \cdot \mathcal{V}_{\sim}^{\nu_v}(\tilde{\pi}, \alpha_n)}{\sum_{\pi \in \Pi} h_{w_n}(\nu_u, \mathcal{U}, \pi)}$$

Now we obtain the

Proof sketch II

Sketch of the proof

Let $\Pi = \{\pi \in \text{Paths}^\omega \mid \pi \models \varphi\}$. To show: $\Pr_{\nu_u, \mathcal{U}}^\omega(\Pi) = \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi)$:

- ① Define scheduler \mathcal{U} to mimic $\mathcal{V}_\sim^{\nu_v}$ on the quotient:

$$\mathcal{U}\left(s_0 \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} s_n\right) := \mathcal{V}_\sim^{\nu_v}\left([s_0] \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} [s_n]\right).$$

- ② Then $\mathcal{U}_\sim^{\nu_u} = \mathcal{V}_\sim^{\nu_v}$ and $\tilde{\nu}_u = \tilde{\nu}_v$:

$$\mathcal{U}_\sim^{\nu_u}(\tilde{\pi}, \alpha_n) = \frac{\sum_{\pi \in \Pi} \text{hw}_n(\nu_u, \mathcal{U}, \pi) \cdot \mathcal{V}_\sim^{\nu_v}(\tilde{\pi}, \alpha_n)}{\sum_{\pi \in \Pi} \text{hw}_n(\nu_u, \mathcal{U}, \pi)}$$

Now we obtain the proof:

$$\boxed{\mathcal{C} : \quad \nu_v, \mathcal{V} \quad \stackrel{!}{=} \quad \nu_u, \mathcal{U}}$$

$$\boxed{\tilde{\mathcal{C}} : \quad \tilde{\nu}_v, \mathcal{V}_\sim^{\nu_v} \quad \stackrel{!}{=} \quad \tilde{\nu}_u, \mathcal{U}_\sim^{\nu_u}}$$

Proof sketch II

Sketch of the proof

Let $\Pi = \{\pi \in \text{Paths}^\omega \mid \pi \models \varphi\}$. To show: $\Pr_{\nu_u, \mathcal{U}}^\omega(\Pi) = \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi)$:

- ① Define scheduler \mathcal{U} to mimic $\mathcal{V}_\sim^{\nu_v}$ on the quotient:

$$\mathcal{U}\left(s_0 \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} s_n\right) := \mathcal{V}_\sim^{\nu_v}\left([s_0] \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} [s_n]\right).$$

- ② Then $\mathcal{U}_\sim^{\nu_u} = \mathcal{V}_\sim^{\nu_v}$ and $\tilde{\nu}_u = \tilde{\nu}_v$:

$$\mathcal{U}_\sim^{\nu_u}(\tilde{\pi}, \alpha_n) = \frac{\sum_{\pi \in \Pi} \text{hw}_n(\nu_u, \mathcal{U}, \pi) \cdot \mathcal{V}_\sim^{\nu_v}(\tilde{\pi}, \alpha_n)}{\sum_{\pi \in \Pi} \text{hw}_n(\nu_u, \mathcal{U}, \pi)}$$

Now we obtain the proof:

$$\Pr_{\nu_v, \mathcal{V}}^\omega(\Pi)$$

$$\boxed{\mathcal{C} : \quad \nu_v, \mathcal{V} \quad \stackrel{!}{=} \quad \nu_u, \mathcal{U}}$$

$$\tilde{\mathcal{C}} : \quad \tilde{\nu}_v, \mathcal{V}_\sim^{\nu_v} \quad \tilde{\nu}_u, \mathcal{U}_\sim^{\nu_u}$$

Proof sketch II

Sketch of the proof

Let $\Pi = \{\pi \in \text{Paths}^\omega \mid \pi \models \varphi\}$. To show: $\Pr_{\nu_u, \mathcal{U}}^\omega(\Pi) = \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi)$:

- ① Define scheduler \mathcal{U} to mimic $\mathcal{V}_\sim^{\nu_v}$ on the quotient:

$$\mathcal{U}\left(s_0 \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} s_n\right) := \mathcal{V}_\sim^{\nu_v}\left([s_0] \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} [s_n]\right).$$

- ② Then $\mathcal{U}_\sim^{\nu_u} = \mathcal{V}_\sim^{\nu_v}$ and $\tilde{\nu}_u = \tilde{\nu}_v$:

$$\mathcal{U}_\sim^{\nu_u}(\tilde{\pi}, \alpha_n) = \frac{\sum_{\pi \in \Pi} \text{hw}_n(\nu_u, \mathcal{U}, \pi) \cdot \mathcal{V}_\sim^{\nu_v}(\tilde{\pi}, \alpha_n)}{\sum_{\pi \in \Pi} \text{hw}_n(\nu_u, \mathcal{U}, \pi)}$$

Now we obtain the proof:

$$\Pr_{\nu_v, \mathcal{V}}^\omega(\Pi) = \sum_{k=0}^{\infty} \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi_k)$$

$$\boxed{\mathcal{C} : \quad \nu_v, \mathcal{V} \quad \stackrel{!}{=} \quad \nu_u, \mathcal{U}}$$

$$\boxed{\tilde{\mathcal{C}} : \quad \tilde{\nu}_v, \mathcal{V}_\sim^{\nu_v} \quad \quad \quad \tilde{\nu}_u, \mathcal{U}_\sim^{\nu_u}}$$

Proof sketch II

Sketch of the proof

Let $\Pi = \{\pi \in \text{Paths}^\omega \mid \pi \models \varphi\}$. To show: $\Pr_{\nu_u, \mathcal{U}}^\omega(\Pi) = \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi)$:

- ① Define scheduler \mathcal{U} to mimic $\mathcal{V}_\sim^{\nu_v}$ on the quotient:

$$\mathcal{U}\left(s_0 \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} s_n\right) := \mathcal{V}_\sim^{\nu_v}\left([s_0] \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} [s_n]\right).$$

- ② Then $\mathcal{U}_\sim^{\nu_u} = \mathcal{V}_\sim^{\nu_v}$ and $\tilde{\nu}_u = \tilde{\nu}_v$:

$$\mathcal{U}_\sim^{\nu_u}(\tilde{\pi}, \alpha_n) = \frac{\sum_{\pi \in \Pi} \text{hw}_n(\nu_u, \mathcal{U}, \pi) \cdot \mathcal{V}_\sim^{\nu_v}(\tilde{\pi}, \alpha_n)}{\sum_{\pi \in \Pi} \text{hw}_n(\nu_u, \mathcal{U}, \pi)}$$

Now we obtain the proof:

$$\Pr_{\nu_v, \mathcal{V}}^\omega(\Pi) = \sum_{k=0}^{\infty} \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi_k) = \sum_{k=0}^{\infty} \Pr_{\tilde{\nu}_v, \mathcal{V}_\sim^{\nu_v}}^\omega(\tilde{\Pi}_k)$$

	$\mathcal{C} : \quad \nu_v, \mathcal{V} \quad \stackrel{!}{=} \quad \nu_u, \mathcal{U}$
	$=$
	$\tilde{\mathcal{C}} : \quad \tilde{\nu}_v, \mathcal{V}_\sim^{\nu_v} \quad \quad \quad \tilde{\nu}_u, \mathcal{U}_\sim^{\nu_u}$

Proof sketch II

Sketch of the proof

Let $\Pi = \{\pi \in \text{Paths}^\omega \mid \pi \models \varphi\}$. To show: $\Pr_{\nu_u, \mathcal{U}}^\omega(\Pi) = \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi)$:

- ① Define scheduler \mathcal{U} to mimic $\mathcal{V}_\sim^{\nu_v}$ on the quotient:

$$\mathcal{U}\left(s_0 \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} s_n\right) := \mathcal{V}_\sim^{\nu_v}\left([s_0] \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} [s_n]\right).$$

- ② Then $\mathcal{U}_\sim^{\nu_u} = \mathcal{V}_\sim^{\nu_v}$ and $\tilde{\nu}_u = \tilde{\nu}_v$:

$$\mathcal{U}_\sim^{\nu_u}(\tilde{\pi}, \alpha_n) = \frac{\sum_{\pi \in \Pi} \text{hw}_n(\nu_u, \mathcal{U}, \pi) \cdot \mathcal{V}_\sim^{\nu_v}(\tilde{\pi}, \alpha_n)}{\sum_{\pi \in \Pi} \text{hw}_n(\nu_u, \mathcal{U}, \pi)}$$

Now we obtain the proof:

$$\begin{aligned} \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi) &= \sum_{k=0}^{\infty} \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi_k) = \sum_{k=0}^{\infty} \Pr_{\tilde{\nu}_v, \mathcal{V}_\sim^{\nu_v}}^\omega(\tilde{\Pi}_k) \\ &= \sum_{k=0}^{\infty} \Pr_{\tilde{\nu}_u, \mathcal{U}_\sim^{\nu_u}}^\omega(\tilde{\Pi}_k) \end{aligned}$$

$$\boxed{\mathcal{C} : \quad \nu_v, \mathcal{V} \quad \stackrel{!}{=} \quad \nu_u, \mathcal{U}}$$

$$\boxed{\tilde{\mathcal{C}} : \quad \tilde{\nu}_v, \mathcal{V}_\sim^{\nu_v} \quad \stackrel{!}{=} \quad \tilde{\nu}_u, \mathcal{U}_\sim^{\nu_u}}$$

Proof sketch II

Sketch of the proof

Let $\Pi = \{\pi \in \text{Paths}^\omega \mid \pi \models \varphi\}$. To show: $\Pr_{\nu_u, \mathcal{U}}^\omega(\Pi) = \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi)$:

- ① Define scheduler \mathcal{U} to mimic $\mathcal{V}_\sim^{\nu_v}$ on the quotient:

$$\mathcal{U}\left(s_0 \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} s_n\right) := \mathcal{V}_\sim^{\nu_v}\left([s_0] \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} [s_n]\right).$$

- ② Then $\mathcal{U}_\sim^{\nu_u} = \mathcal{V}_\sim^{\nu_v}$ and $\tilde{\nu}_u = \tilde{\nu}_v$:

$$\mathcal{U}_\sim^{\nu_u}(\tilde{\pi}, \alpha_n) = \frac{\sum_{\pi \in \Pi} \text{hw}_n(\nu_u, \mathcal{U}, \pi) \cdot \mathcal{V}_\sim^{\nu_v}(\tilde{\pi}, \alpha_n)}{\sum_{\pi \in \Pi} \text{hw}_n(\nu_u, \mathcal{U}, \pi)}$$

Now we obtain the proof:

$$\begin{aligned} \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi) &= \sum_{k=0}^{\infty} \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi_k) = \sum_{k=0}^{\infty} \Pr_{\tilde{\nu}_v, \mathcal{V}_\sim^{\nu_v}}^\omega(\tilde{\Pi}_k) \\ &= \sum_{k=0}^{\infty} \Pr_{\tilde{\nu}_u, \mathcal{U}_\sim^{\nu_u}}^\omega(\tilde{\Pi}_k) = \sum_{k=0}^{\infty} \Pr_{\nu_u, \mathcal{U}}^\omega(\Pi_k) \end{aligned}$$

$$\boxed{\mathcal{C} : \quad \nu_v, \mathcal{V} \quad \stackrel{!}{=} \quad \nu_u, \mathcal{U}}$$

$$\boxed{\tilde{\mathcal{C}} : \quad \tilde{\nu}_v, \mathcal{V}_\sim^{\nu_v} \quad \stackrel{!}{=} \quad \tilde{\nu}_u, \mathcal{U}_\sim^{\nu_u}}$$

Proof sketch II

Sketch of the proof

Let $\Pi = \{\pi \in \text{Paths}^\omega \mid \pi \models \varphi\}$. To show: $\Pr_{\nu_u, \mathcal{U}}^\omega(\Pi) = \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi)$:

- ① Define scheduler \mathcal{U} to mimic $\mathcal{V}_\sim^{\nu_v}$ on the quotient:

$$\mathcal{U}\left(s_0 \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} s_n\right) := \mathcal{V}_\sim^{\nu_v}\left([s_0] \xrightarrow{a_0, t_0} \dots \xrightarrow{a_{n-1}, t_{n-1}} [s_n]\right).$$

- ② Then $\mathcal{U}_\sim^{\nu_u} = \mathcal{V}_\sim^{\nu_v}$ and $\tilde{\nu}_u = \tilde{\nu}_v$:

$$\mathcal{U}_\sim^{\nu_u}(\tilde{\pi}, \alpha_n) = \frac{\sum_{\pi \in \Pi} \text{hw}_n(\nu_u, \mathcal{U}, \pi) \cdot \mathcal{V}_\sim^{\nu_v}(\tilde{\pi}, \alpha_n)}{\sum_{\pi \in \Pi} \text{hw}_n(\nu_u, \mathcal{U}, \pi)}$$

Now we obtain the proof:

$$\begin{aligned} \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi) &= \sum_{k=0}^{\infty} \Pr_{\nu_v, \mathcal{V}}^\omega(\Pi_k) = \sum_{k=0}^{\infty} \Pr_{\tilde{\nu}_v, \mathcal{V}_\sim^{\nu_v}}^\omega(\tilde{\Pi}_k) \\ &= \sum_{k=0}^{\infty} \Pr_{\tilde{\nu}_u, \mathcal{U}_\sim^{\nu_u}}^\omega(\tilde{\Pi}_k) = \sum_{k=0}^{\infty} \Pr_{\nu_u, \mathcal{U}}^\omega(\Pi_k) = \Pr_{\nu_u, \mathcal{U}}^\omega(\Pi). \end{aligned}$$

$$\boxed{\mathcal{C} : \quad \nu_v, \mathcal{V} \quad \stackrel{!}{=} \quad \nu_u, \mathcal{U}}$$

$$\boxed{\tilde{\mathcal{C}} : \quad \tilde{\nu}_v, \mathcal{V}_\sim^{\nu_v} \quad \stackrel{!}{=} \quad \tilde{\nu}_u, \mathcal{U}_\sim^{\nu_u}}$$

Related Work

Measure theoretic basis of THR schedulers

[Wolovick et al., FORMATS 06]

Probabilistic branching time logics

[Baier et al., Distr. Comp. 98]

Long run average behaviour

[de Alfaro, LICS 1998]

Time-bounded reachability in uniform CTMDPs

[Baier et al., TCS 05]

Model Checking of prob. and nondet. systems

[Bianco et al., FSTTCS 95]

Abstraction for continuous-time Markov chains

[Katoen et al., CAV 07]

Summary

What did we do?

- ① provided the measure-theoretic foundations of CSL on CTMDPs,
- ② defined strong bisimulation on CTMDPs and
- ③ proved the preservation of CSL under strong bisimilarity

Example: Model minimization

Bisimulation minimization preserves functional and observational invariance

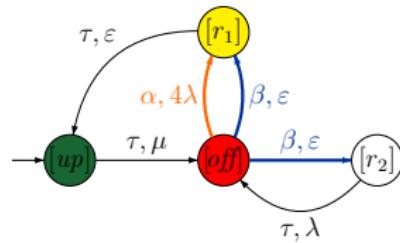
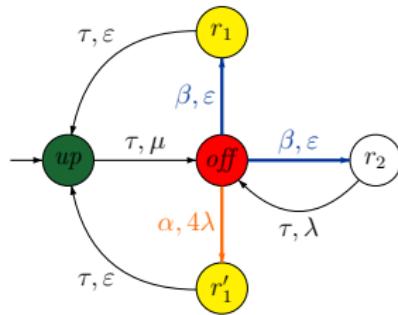
Summary

What did we do?

- ① provided the measure-theoretic foundations of CSL on CTMDPs,
- ② defined strong bisimulation on CTMDPs and
- ③ proved the preservation of CSL under strong bisimilarity

Example (Model minimization)

Bisimulation minimization preserves transient and steady-state measures:



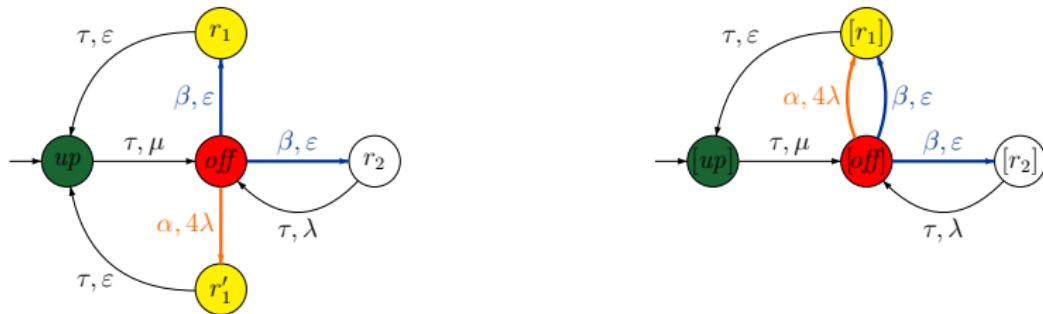
Summary

What did we do?

- ① provided the measure-theoretic foundations of CSL on CTMDPs,
- ② defined strong bisimulation on CTMDPs and
- ③ proved the preservation of CSL under strong bisimilarity

Example (Model minimization)

Bisimulation minimization preserves transient and steady-state measures:



Thank you for your attention!