# Bisimulation and Logical Preservation for Continuous-Time Markov Decision Processes

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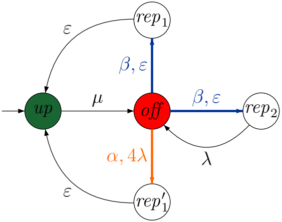
# Repairman's Roulette

## Maintenance of a safety critical system:

- System is operational in state $up$.
- Failures are exponentially distributed:
  - Mean delay to next failure $\frac{1}{\mu}$
  - Probability for a failure within time $t$.




- A failure moves the system from $up$ to $off$.
- Two types of repairmen:
  1. **cautious**: slow and reliable via $\alpha$ or
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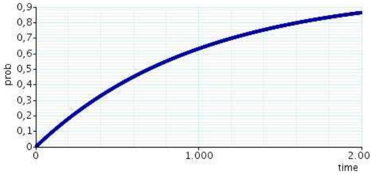


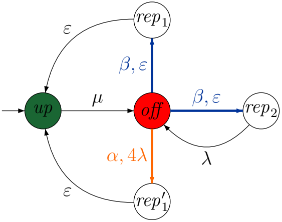


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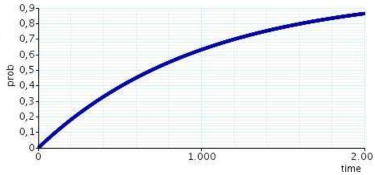


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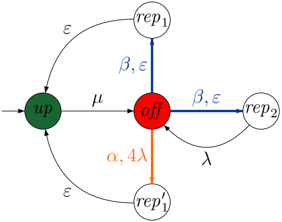

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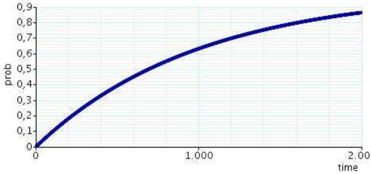


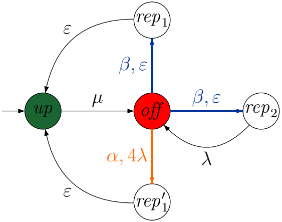


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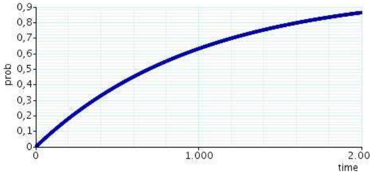


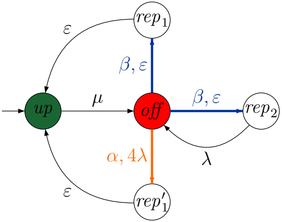


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**Measure of interest:** Probability to return to $up$ in $t$ time units after a failure.

## Why Continuous-Time Markov Decision Processes?

1. CTMDPs are an important model in
   - stochastic control theory [Qiu et al.]
   - stochastic scheduling [Feinberg et al.]
2. CTMDPs provide the semantic basis for
   - non-well-specified stochastic activity networks [Sanders et al.]
   - generalised stochastic Petri nets with confusion [Chiola et al.]
   - Markovian process algebras [Hermanns et al., Hillston et al.]

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3. Adaptation of Continuous Stochastic Logic (cf. CTL) to CTMDPs.
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3. Adaptation of **Continuous Stochastic Logic** (cf. CTL) to CTMDPs.
4. **Preservation** of CSL under strong bisimulation.

## Definition (Continuous Time Markov Decision Process)

A CTMDP $(\mathcal{S}, \mathrm{Act}, \mathbf{R}, \mathrm{AP}, \mathrm{L})$ has a finite set of states $\mathcal{S}$ and propositions $\mathrm{AP}$. $\mathrm{L} : \mathcal{S} \to 2^{\mathrm{AP}}$ is its state labelling. Further

- $\mathrm{Act} = \{\alpha, \beta, \dots\}$ is a finite set of actions and
- $\mathbf{R} : \mathcal{S} \times \mathrm{Act} \times \mathcal{S} \to \mathbb{R}_{\geq 0}$ is a transition rate matrix such that
  - $\mathbf{R}(s, \alpha, s') = \lambda$ is the rate of a negative exponential distribution

$$f_X(t) = \begin{cases} \lambda \cdot e^{-\lambda \cdot t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad E[X] = \frac{1}{\lambda}$$

  such that $\mathrm{Act}(s) = \{\alpha \in \mathrm{Act} \mid \exists s' \in \mathcal{S}.\ \mathbf{R}(s, \alpha, s') > 0\} \neq \emptyset$ for all $s \in \mathcal{S}$.
  - $E(s, \alpha) = \sum_{s' \in \mathcal{S}} \mathbf{R}(s, \alpha, s')$ is the **exit rate** of $s$ under $\alpha$.

## Example

1. Choose action in $\mathrm{Act}(s_1) = \{\alpha, \beta\}$ **nondeterministically**: say $\alpha$
2. **Race condition** between $\alpha$-transitions:
   - Mean delay: $\frac{1}{\mathbf{R}(s_1,\alpha,s_2)+\mathbf{R}(s_1,\alpha,s_3)} = \frac{1}{20}$
   - Probability to move to $s_3$: $\frac{\mathbf{R}(s_1,\alpha,s_3)}{E(s,\alpha)} = \frac{1}{4}$

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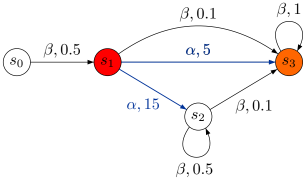

# Transitions and paths

## Definition (Paths)

1. **Finite paths** of length $n \in \mathbb{N}$ are denoted

$$\pi = s_0 \xrightarrow{\alpha_0, t_0} s_1 \xrightarrow{\alpha_1, t_1} s_2 \xrightarrow{\alpha_2, t_2} \cdots \xrightarrow{\alpha_{n-1}, t_{n-1}} s_n$$

   - $\mathrm{Paths}^n$ is the set of paths of length $n$ and
   - $\pi\!\downarrow = s_n$ is the last state of $\pi$.

2. $\mathrm{Paths}^\omega$ is the corresponding set of **infinite paths**
   - $\pi@t$ is the state occupied at time $t$ on path $\pi$.

## Definition (Combined transition)

A combined transition $m = (\alpha, t, s')$

- $\alpha$ is the action (chosen externally),
- $t$ is the transition's firing time and
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# Definition (Measurable spaces)

Probability measures are defined on $\sigma$-fields

1. $\mathfrak{F}$ of sets of combined transitions:

$$\Omega := \mathrm{Act} \times \mathbb{R}_{\geq 0} \times \mathcal{S}$$

$$\mathfrak{F} := \sigma\big(\mathfrak{F}_{\mathrm{Act}} \times \mathfrak{B}(\mathbb{R}_{\geq 0}) \times \mathfrak{F}_{\mathcal{S}}\big)$$

$\mathfrak{B}(\mathbb{R}_{\geq 0})$ : Borel $\sigma$-field for $\mathbb{R}_{\geq 0}$

2. $\mathfrak{F}_{\mathrm{Paths}^n}$ of sets of paths of length $n$:

$$\mathfrak{F}_{\mathrm{Paths}^n} := \sigma\left(\{S_0 \times M_1 \times \cdots \times M_n \mid S_0 \in \mathfrak{F}_{\mathcal{S}}, M_i \in \mathfrak{F}\}\right)$$

3. $\mathfrak{F}_{\mathrm{Paths}^\omega}$ of sets of infinite paths:

Cylinder set construction:

- Any $C^n \in \mathfrak{F}_{\mathrm{Paths}^n}$ defines a cylinder base (of finite length)
- $C_n := \{\pi \in \mathrm{Paths}^\omega \mid \pi[0..n] \in C^n\}$ is a cylinder (extension to infinity).

The $\sigma$-field $\mathfrak{F}_{\mathrm{Paths}^\omega}$ is then

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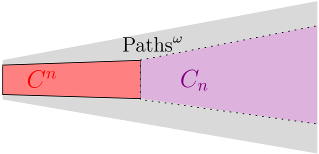

## Defining probability measures

Schedulers resolve nondeterministic choices between actions. Classes are

1. either **deterministic** or **randomized** and
2. depending on available **history**.

### Definition (Measurable scheduler)

A measurable scheduler [WJ, 2006] is a mapping

$$\mathcal{D} : \mathrm{Paths}^{*} \times \mathfrak{F}_{\mathrm{Act}} \rightarrow [0,1] \quad \text{where:}$$

1. $\mathcal{D}(\pi, \cdot) \in \mathrm{Distr}(\mathrm{Act}(\pi\downarrow))$ for $\pi \in \mathrm{Paths}^{*}$
2. $\mathcal{D}(\cdot, A)$ are measurable for $A \in \mathfrak{F}_{\mathrm{Act}}$.

### Example (Why such schedulers?)

Stationary schedulers are not optimal:

- $t \leq t_0$: choose $\beta$ to maximize probability
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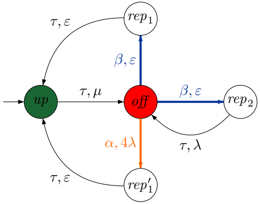


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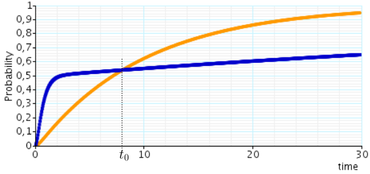


# Example (One-step probabilities)

The event to go from *off* to $rep_1$ in $2$ time units:

$$M = \{\beta\} \times [0, 2] \times \{rep_1\} \in \mathfrak{F}.$$

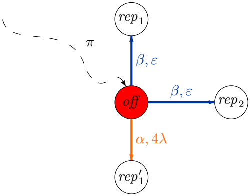
Its probability $\mu_{\mathcal{D}}(\pi, M)$ depends on:

1. the probability $\mathcal{D}(\pi, \{\beta\})$ of choosing $\beta$
2. the exp. distr. residence time in *off*:

$$\eta_{E(off,\beta)}(t) = E(off,\beta) \cdot e^{-E(off,\beta)\cdot t} = 2\varepsilon \cdot e^{-2\varepsilon \cdot t}$$

3. the race between $rep_1$ and $rep_2$:

$$\mathbf{P}(off, \beta, rep_1) = \frac{\mathbf{R}(off, \beta, rep_1)}{E(off, \beta)} = \frac{\varepsilon}{2\varepsilon} = 0.5.$$



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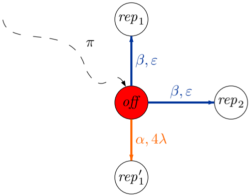

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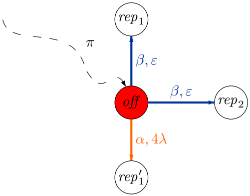

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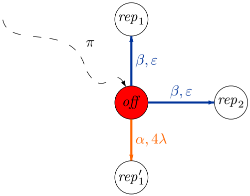


## Definition (Measuring sets of combined transitions)

For history $\pi \in \mathrm{Paths}^\star$ and $\mathcal{D} \in \mathrm{THR}$ define **probability measure** $\mu_{\mathcal{D}}(\pi, \cdot)$ on $\mathfrak{F}$:

$$\mu_{\mathcal{D}}(\pi, M) := \int_{\mathrm{Act}} \mathcal{D}(\pi, d\alpha) \int_{\mathbb{R}_{\geq 0}} \eta_{E(\pi\downarrow,\alpha)}(dt) \int_{\mathcal{S}} \mathbf{I}_M(\alpha, t, s) \;\; \mathbf{P}(\pi\downarrow, \alpha, ds).$$

# Definition (Measuring sets of paths)

1. Initial distribution $\nu$:
   The probability to start in state $s$ is $\nu(\{s\}) = \nu(s)$.

2. $Pr^n_{\nu,\mathcal{D}}$ on sets of finite paths:
   Let $\nu \in Distr(S)$ and $\mathcal{D} \in THR$. Define inductively:

$$Pr^0_{\nu,\mathcal{D}}(\Pi) := \sum_{s \in \Pi} \nu(s)$$

$$Pr^{n+1}_{\nu,\mathcal{D}}(\Pi) := \int_{Paths^n} Pr^n_{\nu,\mathcal{D}}(d\pi) \int_{\Omega} \mathbf{I}_{\Pi}(\pi \circ m)\, \mu_{\mathcal{D}}(\pi, dm)$$

3. $Pr^\omega_{\nu,\mathcal{D}}$ on sets of infinite paths:
   - A cylinder base is a measurable set $C^n \in \mathfrak{F}_{Paths^n}$
   - $C^n$ defines cylinder $C_n = \{\pi \in Paths^\omega \mid \pi[0..n] \in C^n\}$
   - The probability of cylinder $C_n$ is that of its base $C^n$:

$$Pr^\omega_{\nu,\mathcal{D}}(C_n) = Pr^n_{\nu,\mathcal{D}}(C^n).$$

   This extends to $\mathfrak{F}_{Paths^\omega}$ by Ionescu-Tulcea.

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   The probability to start in state $s$ is $\nu(\{s\}) = \nu(s)$.

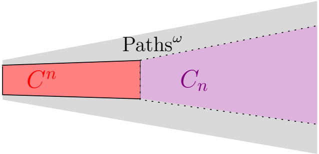
2. $\Pr^n_{\nu,\mathcal{D}}$ on sets of finite paths:
   Let $\nu \in \mathrm{Distr}(\mathcal{S})$ and $\mathcal{D} \in \mathrm{THR}$. Define inductively:

$$\Pr^0_{\nu,\mathcal{D}}(\Pi) := \sum_{s \in \Pi} \nu(s)$$

$$\Pr^{n+1}_{\nu,\mathcal{D}}(\Pi) := \int_{\mathrm{Paths}^n} \Pr^n_{\nu,\mathcal{D}}(d\pi) \int_{\Omega} \mathbf{I}_{\Pi}(\pi \circ m)\, \mu_{\mathcal{D}}(\pi, dm)$$

3. $\Pr^\omega_{\nu,\mathcal{D}}$ on sets of infinite paths:
   - A **cylinder base** is a measurable set $C^n \in \mathfrak{F}_{\mathrm{Paths}^n}$
   - $C^n$ defines **cylinder** $C_n = \{\pi \in \mathrm{Paths}^\omega \mid \pi[0..n] \in C^n\}$
   - The probability of cylinder $C_n$ is that of its base $C^n$:

$$\Pr^\omega_{\nu,\mathcal{D}}(C_n) = \Pr^n_{\nu,\mathcal{D}}(C^n).$$




This extends to $\mathfrak{F}_{\mathrm{Paths}^\omega}$ by **Ionescu-Tulcea**.

## Definition (Strong bisimulation relation)

Equivalence $\mathcal{R} \subseteq \mathcal{S} \times \mathcal{S}$ is a **strong bisimulation relation** iff for all $(u, v) \in \mathcal{R}$ and all $\alpha \in \mathrm{Act}$:

1. $L(u) = L(v)$ and
2. $\forall C \in \mathcal{S}_{\mathcal{R}}.\ \mathbf{R}(u, \alpha, C) = \mathbf{R}(v, \alpha, C)$.

$\mathbf{R}(u, \alpha, C) := \sum_{u' \in C} \mathbf{R}(u, \alpha, u')$

$u, v$ are **strongly bisimilar** ($u \sim v$) iff $\exists$ str. bisim. relation $\mathcal{R}$ with $(u, v) \in \mathcal{R}$.

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## Definition (Quotient under $\sim$)

For $\mathcal{C} = (\mathcal{S}, \mathrm{Act}, \mathbf{R}, \mathrm{AP}, \mathrm{L})$, its **quotient under** $\sim$ is $\tilde{\mathcal{C}} := (\tilde{\mathcal{S}}, \mathrm{Act}, \tilde{\mathbf{R}}, \mathrm{AP}, \tilde{\mathrm{L}})$:

$$\tilde{\mathrm{L}}\left([s]\right) := \mathrm{L}(s)$$

$$\tilde{\mathbf{R}}\left([s], \alpha, C\right) := \mathbf{R}(s, \alpha, C)$$

$\tilde{\mathcal{S}} := \{[s]_\sim \mid s \in \mathcal{S}\}$

Example

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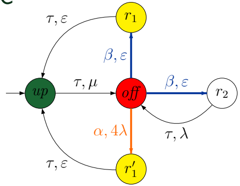
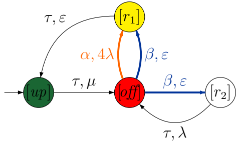


# Continuous Stochastic Logic [Aziz et al. 2000, Baier et al. 2003]

## Example (Transient state formula)

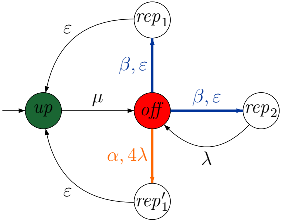
In state $\mathit{off}$, the probability to reach the $up$ state within 20 time units exceeds $0.5$ under any scheduling decision:

$$\Phi = \mathit{off} \rightarrow \forall^{>0.5} \Diamond^{[0,20]} up$$

## Example (Long-run average [de Alfaro, LICS 98])

For any scheduler, the system on average is not operational for less than $1\%$ of its execution time:

$$\Psi = \mathsf{L}^{<0.01} \neg up$$



## Definition (Syntax of CSL)

For $a \in \mathrm{AP}$, $p \in [0,1]$, $I \subseteq \mathbb{R}_{\geq 0}$ a nonempty interval and $\sqsubseteq \in \{<, \leq, \geq, >\}$, **state formulas** and **path formulas** are:

$$\Phi ::= \; a \mid \neg\Phi \mid \Phi \wedge \Phi \mid \forall^{\sqsubseteq p}\varphi \mid \mathsf{L}^{\sqsubseteq p}\Phi \qquad\qquad \varphi ::= \; \mathsf{X}^{I}\Phi \mid \Phi \mathsf{U}^{I} \Phi.$$

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## Definition (Semantics of path formulas)

For CTMDP $\mathcal{C}$ and infinite path $\pi = s_0 \xrightarrow{\alpha_0, t_0} s_1 \xrightarrow{\alpha_1, t_1} \cdots$ define:

$$\pi \models \mathsf{X}^I\Phi \iff \pi[1] \models \Phi \wedge t_0 \in I$$

$$\pi \models \Phi\mathsf{U}^I\Psi \iff \exists t \in I.\ (\pi@t \models \Psi \wedge (\forall t' \in [0,t).\ \pi@t' \models \Phi))\,.$$

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Let $\varphi = \Phi\mathsf{U}^{[1,2]}\Psi$ and $\pi \in Paths^\omega$ as follows:

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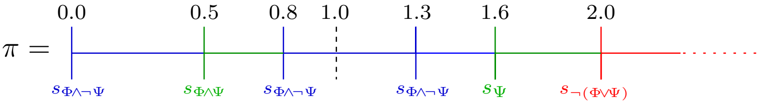

## Average residence time

For state formula $\Phi$, path $\pi = s_0 \xrightarrow{\alpha_0, t_0} s_1 \xrightarrow{\alpha_1, t_1} s_2 \cdots$ and time point $t$:

$$\mathrm{avg}(\Phi, t, \pi) := \frac{1}{t} \int_0^t \mathbf{I}_{\mathrm{Sat}(\Phi)}(\pi@t')\, dt'.$$

$$\mathbf{I}_{\mathrm{Sat}(\Phi)}(s) := \begin{cases} 1 & \text{if } s \in \mathrm{Sat}(\Phi) \\ 0 & \text{otherwise} \end{cases}$$

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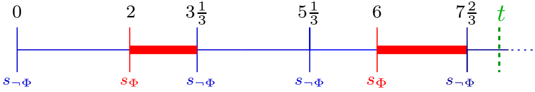
## Example

The average time spent in $\mathrm{Sat}(\Phi)$-states up to time $t = 8$:

$\pi =$ 


$$\mathrm{avg}(\Phi, 8, \pi) = \frac{\frac{4}{3} + \frac{5}{3}}{8} = \frac{3}{8}$$

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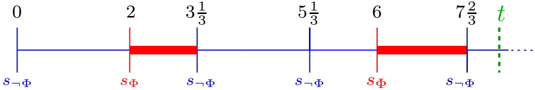
$\pi =$ 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## Definition (Semantics of state formulas)

Let $p \in [0,1]$, $\sqsubseteq \in \{\leq, <, \geq, >\}$ and $\Phi, \varphi$ state and path formulas:

$$s \models \mathsf{L}^{\sqsubseteq p}\Phi \iff \forall D \in \mathrm{THR}.\ \lim_{t \to \infty} \int_{\mathrm{Paths}^\omega} \operatorname{avg}(\Phi, t, \pi)\ \Pr^\omega_{\nu_s, \mathcal{D}}(d\pi)\ \sqsubseteq p$$

$$s \models \forall^{\sqsubseteq p}\varphi \iff \forall \mathcal{D} \in \mathrm{THR}.\ \Pr^\omega_{\nu_s, \mathcal{D}}\{\pi \in \mathrm{Paths}^\omega \mid \pi \models \varphi\} \sqsubseteq p$$

# Strong bisimilarity preserves CSL properties

**Theorem**

*Let $(\mathcal{S}, \mathrm{Act}, \mathbf{R}, \mathrm{AP}, \mathrm{L})$ be a CTMDP. For all states $u, v \in \mathcal{S}$ it holds:*

$$u \sim v \implies \forall \Phi \in \mathrm{CSL}.\ (u \models \Phi \iff v \models \Phi)\,.$$

*Proof by structural induction:*

1. $a$, $\neg$, $\wedge$ **omitted**
2. $\forall^{\sqsubseteq p}\varphi$ **sketch in this talk**
3. $\mathsf{L}^{\sqsubseteq p}\varphi$ **straightforward, relies on** $\forall^{\sqsubseteq p}\varphi$

# The reverse conjecture does not hold:

## Counterexample [Segala et al., Nordic J. of Comp. 1995, Baier 1998]


- $s \not\sim s'$ as $\mathbf{R}(s, \beta, [u_2]) = 4$ whereas $\mathbf{R}(s', \beta, \cdot) = 0$.
- For all $\Phi$ it holds: $s \models \Phi \Longleftrightarrow s' \models \Phi$ :
  - CSL can only express extreme probability bounds.
  - The choice of action $\beta$ does not contribute an infimum or supremum.
  - Some examples:
    - $\Phi = \exists^{\leq \frac{1}{3}} X^{[0,\infty)}$ *yellow*: choose $\beta$
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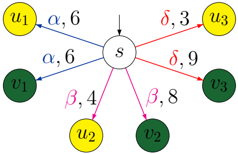


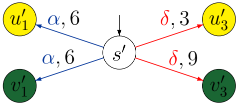


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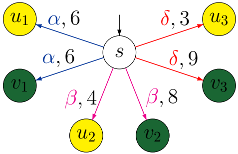


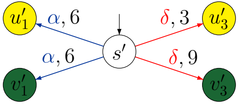


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$\exists^{\sqsubseteq p} \varphi \equiv \neg \forall^{\sqsupset p} \varphi$

# Preservation of CSL formulas

## Proof idea for transient state formulas

- Assume $u \sim v$ and $u \models \forall^{\sqsubseteq p} \varphi$. To prove: $v \models \forall^{\sqsubseteq p} \varphi$, i.e.

$$\forall \mathcal{V} \in \mathrm{THR}.\ \Pr^{\omega}_{\nu_v, \mathcal{V}} \{\pi \in \mathrm{Paths}^{\omega} \mid \pi \models \varphi\} \sqsubseteq p. \qquad (1)$$

- Let $\mathcal{V} \in \mathrm{THR}$ be a scheduler (w.r.t. $\nu_v$).
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For a specific subclass of sets of paths:
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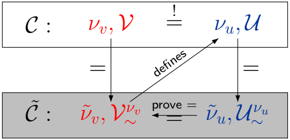

# Preliminaries

## Definition (Simple bisimulation closed sets of paths)

A measurable set of paths of the form

$$\Pi = [s_0] \times A_0 \times T_0 \times [s_1] \times \cdots \times A_{n-1} \times T_{n-1} \times [s_n]$$

is called **simple bisimulation closed**. $\Pi$ corresponds to the set $\tilde{\Pi}$ on $\tilde{\mathcal{C}}$:

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## Preliminary goal

For initial distribution $\nu \in \mathrm{Distr}(S)$ and scheduler $\mathcal{D} \in \mathrm{THR}$:
Provide a **measure preserving** scheduler $\mathcal{D}^{\nu}_{\sim}$ on $\tilde{\mathcal{C}}$ such that

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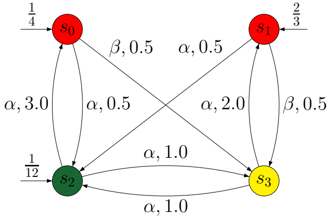
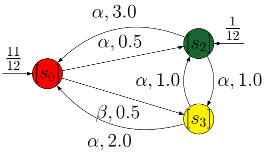
# Example




Let the first decision of $\mathcal{D}$ be as follows:

Intuitively, the quotient scheduler $\mathcal{D}^{\nu}_{\sim}$ then decides in $[s_0]$ as follows:

$$\mathcal{D}^{\nu}_{\sim}([s_0], \{\alpha\}) = \frac{\sum_{s\in[s_0]} \nu(s)\cdot\mathcal{D}(s,\{\alpha\})}{\sum_{s\in[s_0]}\nu(s)} = \frac{\frac{1}{4}\cdot\frac{2}{3}+\frac{2}{3}\cdot\frac{1}{4}}{\frac{1}{4}+\frac{2}{3}} = \frac{4}{11}$$

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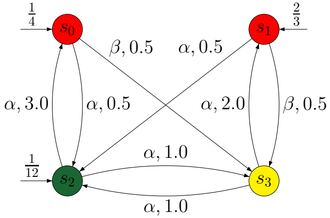
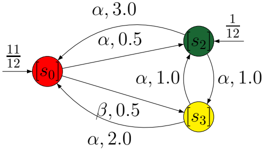
# Example




Let the first decision of $\mathcal{D}$ be as follows:

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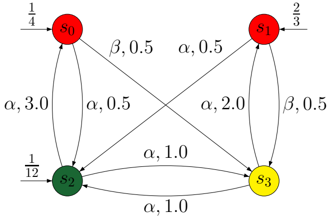


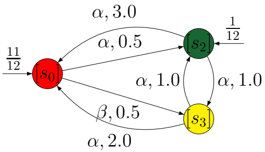


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# Quotient scheduler

## Definition (History weight)

Given CTMDP $\mathcal{C}$, $\nu \in \mathrm{Distr}(\mathcal{S})$ and $\mathcal{D} \in \mathrm{THR}$.
Define the weight of history $\pi \in \mathrm{Paths}^{\star}$ inductively:

$$hw_0(\nu, \mathcal{D}, \pi) := \nu(\pi) \quad \text{if } \pi \in \mathrm{Paths}^0 = \mathcal{S} \text{ and}$$

$$hw_{n+1}(\nu, \mathcal{D}, \pi \xrightarrow{\alpha_n, t_n} s_{n+1}) := hw_n(\nu, \mathcal{D}, \pi) \cdot \mathcal{D}(\pi, \{\alpha_n\}) \cdot \mathbf{P}(\pi{\downarrow}, \alpha_n, s_{n+1}).$$

## Definition (Quotient scheduler)

For any history $\bar{\pi} = [s_0] \xrightarrow{\alpha_0, t_0} [s_1] \xrightarrow{\alpha_1, t_1} \cdots \xrightarrow{\alpha_{n-1}, t_{n-1}} [s_n]$, the quotient scheduler $\mathcal{D}^{\nu}_{\sim}$ is defined by:

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For any history $\tilde{\pi} = [s_0] \xrightarrow{\alpha_0, t_0} [s_1] \xrightarrow{\alpha_1, t_1} \cdots \xrightarrow{\alpha_{n-1}, t_{n-1}} [s_n]$, the quotient scheduler $\mathcal{D}^\nu_\sim$ is defined by:

$$\mathcal{D}^\nu_\sim(\tilde{\pi}, \{\alpha_n\}) := \frac{\sum_{\pi \in \Pi} hw_n(\nu, \mathcal{D}, \pi) \cdot \mathcal{D}(\pi, \{\alpha_n\})}{\sum_{\pi \in \Pi} hw_n(\nu, \mathcal{D}, \pi)}$$

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# Proof sketch I

## Lemma

1. *For* **simple bisimulation closed** *sets of paths* $\Pi$*:*

$$\Pr^{\omega}_{\nu,\mathcal{D}}(\Pi) = \Pr^{\omega}_{\tilde{\nu},\mathcal{D}^{\nu}_{\sim}}(\tilde{\Pi}).$$

2. *For any path formula* $\varphi$ *there exists a family* $\{\Pi_k\}_{k\in\mathbb{N}}$ *such that*

$$\left\{\pi \in \mathrm{Paths}^{\omega} \;\middle|\; \pi \models \varphi\right\} = \biguplus_{k=0}^{\infty} \Pi_k$$

*and* $\Pi_k$ *is simple bisimulation closed.*

## What have we gained?

For any $\Pi_k \subseteq \{\pi \in \mathrm{Paths}^{\omega} \mid \pi \models \varphi\}$:

$$\Pr^{\omega}_{\nu,\mathcal{D}}(\Pi_k) = \Pr^{\omega}_{\tilde{\nu},\mathcal{D}^{\nu}_{\sim}}(\tilde{\Pi}_k).$$

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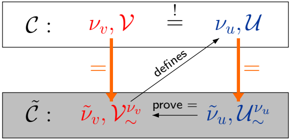

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Let $\Pi = \{\pi \in \text{Paths}^\omega \mid \pi \models \varphi\}$. To show: $\Pr^{\omega}_{\nu_u,\mathcal{U}}(\Pi) = \Pr^{\omega}_{\nu_v,\mathcal{V}}(\Pi)$:

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| | | | |
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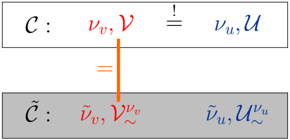

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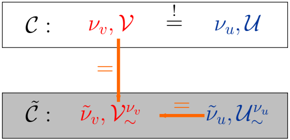

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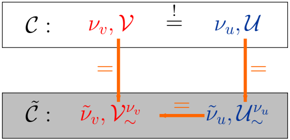

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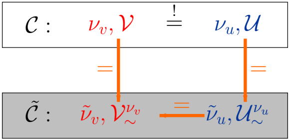

## Related Work

| | |
|---|---|
| Measure theoretic basis of THR schedulers | [Wolovick et al., FORMATS 06] |
| Probabilistic branching time logics | [Baier et al., Distr. Comp. 98] |
| Long run average behaviour | [de Alfaro, LICS 1998] |
| Time-bounded reachability in uniform CTMDPs | [Baier et al., TCS 05] |
| Model Checking of prob. and nondet. systems | [Bianco et al., FSTTCS 95] |
| Abstraction for continuous-time Markov chains | [Katoen et al., CAV 07] |

# Summary

## What did we do?

1. provided the measure-theoretic foundations of CSL on CTMDPs,
2. defined strong bisimulation on CTMDPs and
3. proved the preservation of CSL under strong bisimilarity

## Example (Model minimization)

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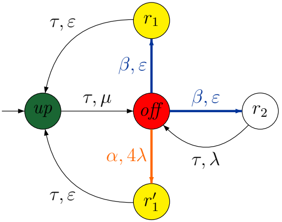


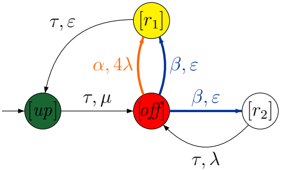

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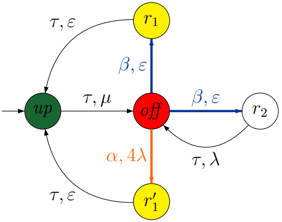
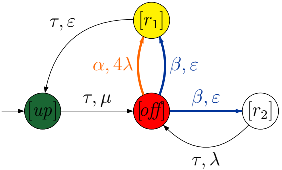
## Example (Model minimization)

Bisimulation minimization preserves transient and steady-state measures:




**Thank you for your attention!**