From Measure Theory to CTMDPs

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Measure Theory

Our Setting

Assume a set Ω, called **sample space**.
Subsets A of Ω are called **events**.
Idea: Measure the {size | probability | volume | length} of events!

Intuition: Let $\omega \in \Omega$ be the outcome of an experiment.
Then A is an event if $\omega \in A$ can be decided.

Fields and σ–fields

Definition (Field)

A class of subsets $\mathfrak{F}$ of Ω is a field iff

1. $\Omega \in \mathfrak{F}$.
2. $A \in \mathfrak{F} \Rightarrow A^c \in \mathfrak{F}$.
3. $A_1, \ldots, A_n \in \mathfrak{F} \implies \bigcup_{i=1}^{n} A_i \in \mathfrak{F}$ where $n \in \mathbb{N}$

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$\mathfrak{F}$ is a σ**–field** iff it is closed under countable union:

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Let $\mathcal{C} \subseteq 2^{\Omega}$. $\sigma(\mathcal{C})$ denotes the **smallest σ–field** containing $\mathcal{C}$.

Example: The Borel σ–field

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Define the class of **finite disjoint unions** of right–semiclosed intervals:

$$\mathfrak{F}_0(\mathbb{R}) := \{I_1 \uplus I_2 \uplus \cdots \uplus I_n \mid n \in \mathbb{N}\}.$$

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$\mathfrak{F}_0(\mathbb{R})$ is a field.

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Borel σ–field

Let $\mathcal{E} = \{(a, b] \mid a, b \in \mathbb{R}$ and $a < b\}$ and
let $\sigma(\mathcal{E})$ denote the **smallest σ–field** containing $\mathcal{E}$.
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$\mathfrak{B}(\mathbb{R})$ has many generators:

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Intuition: Construct σ–field by forming countable unions and complements of intervals in all possible ways.

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Let $\mathfrak{F}$ be a σ–field over subsets of Ω. A **measure** is a function

$$\mu : \mathfrak{F} \to \bar{\mathbb{R}}_{\geqslant 0} \qquad \text{where } \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$$

which is **countably additive**:

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu(A_i) \qquad \text{for disjoint sets } A_i \in \mathfrak{F}.$$

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Remark: If $\mu(\Omega) = 1$, μ is a **probability measure**.

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On the set of **finite disjoint unions** of right–semiclosed intervals:
Let $I_1 \uplus \cdots \uplus I_n \in \mathfrak{F}_0(\mathbb{R})$. Extend μ to $\mathfrak{F}_0(\mathbb{R})$ by defining

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But: What about $\mu(A)$ for **arbitrary** $A \in \mathfrak{B}(\mathbb{R})$?

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Theorem (Carathéodory Extension Theorem)

Let $\mathfrak{F}_0$ be a field over subsets of a set Ω and let μ be a **measure** *on $\mathfrak{F}_0$. If μ is σ–finite, i.e.*

$$\Omega = \bigcup_{i=1}^{\infty} A_i \qquad \text{where } A_i \in \mathfrak{F}_0 \text{ and } \mu(A_i) < \infty,$$

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In practice: Avoid the σ–field whenever possible!

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$$\mu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j)$$

where $A_1, A_2, \dots \in \mathfrak{F}_0(\mathbb{R})$, $\bigcup_{j=1}^{\infty} A_j \in \mathfrak{F}_0$ and the A_j disjoint.

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Theorem

Let $F : \mathbb{R} \to \mathbb{R}$ be a distrib. function. Let $\mu(a, b] := F(b) - F(a)$. There is a unique extension of μ to a Lebesgue–Stieltjes measure on $\mathbb{R}$.

Thus: Countable additivity of μ follows by defining $F(x) := x$.

Lebesgue's Intuition

Lebesgue about his integral

"One might say that Riemann's approach is comparable to a messy merchant who counts coins in the order they come to his hand whereas we act like a prudent merchant who says:

- I have A_1 coins à one crown, that is $A_1 \cdot 1$ crowns,
- A_2 coins à two crowns, that is $A_2 \cdot 2$ crowns and
- A_3 coins à five crowns, that is $A_3 \cdot 5$ crowns.

Therefore I have $A_1 \cdot 1 + A_2 \cdot 2 + A_3 \cdot 5$ crowns.

Both approaches – no matter how rich the merchant might be – lead to the same result since he only has to count a finite number of coins.
But for us who must add infinitely many indivisibles, the difference between the approaches is essential."

H.Lebesgue, 1926

Measurability

Definition (Measurability)

Let Ω_1, Ω_2 be sets with associated σ–fields $\mathfrak{F}_1$ and $\mathfrak{F}_2$.
$h : \Omega_1 \to \Omega_2$ is **measurable** iff

$$h^{-1}(A) \in \mathfrak{F}_1 \qquad \text{for each } A \in \mathfrak{F}_2$$

Notation: $h : (\Omega_1, \mathfrak{F}_1) \to (\Omega_2, \mathfrak{F}_2)$.

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Some remarks:

- h is **Borel measurable** if $h : (\Omega, \mathfrak{F}) \to (\bar{\mathbb{R}}, \mathfrak{B}(\bar{\mathbb{R}}))$.

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- In probability theory, h is called a **random variable**.

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If h is a **simple** function, it can be represented as

$$h(\omega) := \sum_{i=1}^{n} x_i \cdot \mathbf{I}_{A_i}(\omega)$$

where $A_i \in \mathfrak{F}$ are pairwise disjoint.

$\mathbf{I}_{A_i}$ denotes the indicator function $\mathbf{I}_{A_i}(\omega) := \begin{cases} 1 & \text{if } \omega \in A_i \\ 0 & \text{otherwise} \end{cases}$.

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Intuition: Choose A_i as the **preimage** of x_i!

Lebesgue Integral

Definition (Lebesgue Integral for Simple Functions)

Let $(\Omega, \mathfrak{F}, \mu)$ be a measure space, $h : \Omega \to \bar{\mathbb{R}}$ simple:

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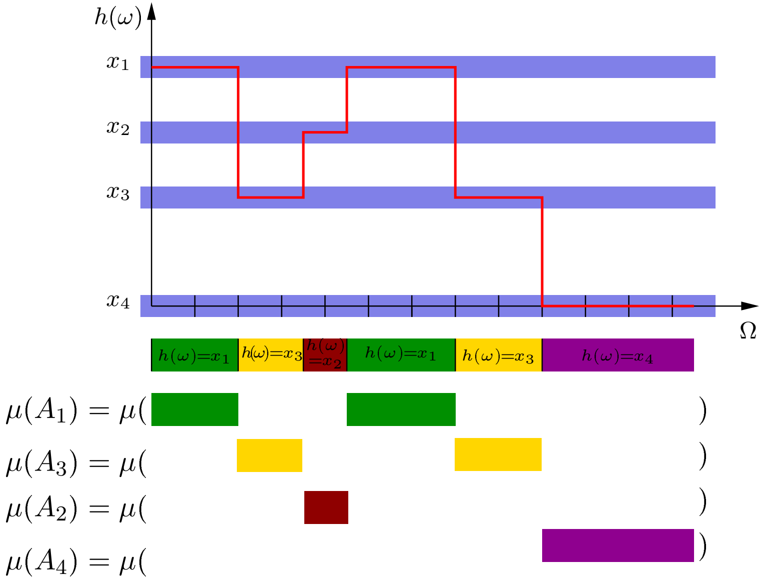
The **Lebesgue–integral** of h is defined as

$$\int_\Omega h \, d\mu := \sum_{i=1}^{n} x_i \cdot \mu(A_i).$$

Intuition: Multiply each x_i with the measure of its preimage A_i.

Example: Lebesgue Integral



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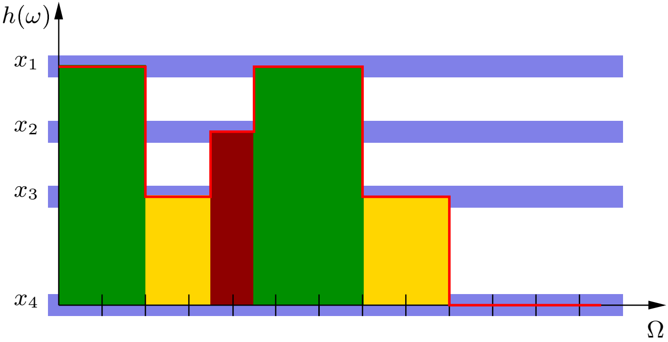
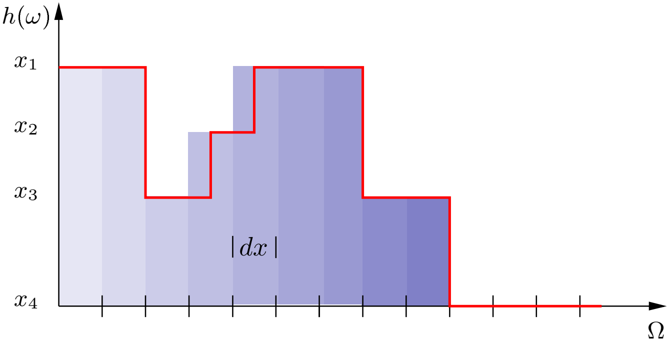


$$\int_\Omega h \, d\mu = x_1\mu(A_1) + x_2\mu(A_2) + x_3\mu(A_3)$$

Example: Riemann (Darboux) Integral



Lebesgue Integral on Nonnegative Functions

Definition

If h is **nonnegative Borel measurable**, then

$$\int_\Omega h \, d\mu := \sup \left\{ \int_\Omega s \, d\mu \mid s \text{ is simple and } 0 \leqslant s \leqslant h \right\}.$$

Theorem

A nonnegative Borel measurable function h is the limit of an increasing sequence of nonnegative simple functions h_n.

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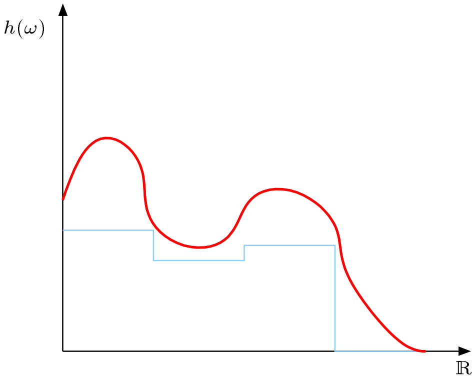

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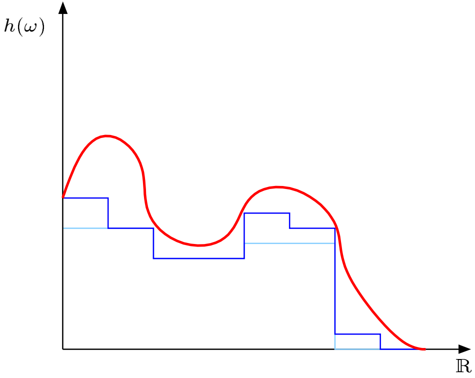

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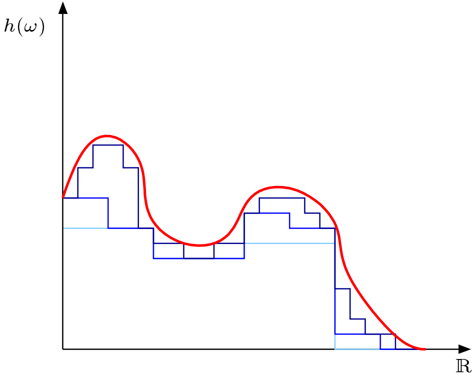

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- $\Omega = \Omega_1 \times \cdots \times \Omega_n$
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- The **product σ–field** $\mathfrak{F}$ is the smallest σ–field containing all measurable rectangles:

$$\mathfrak{F} := \sigma\Big(\mathfrak{F}_1 \times \mathfrak{F}_2 \times \cdots \times \mathfrak{F}_n\Big)$$

Measures on Finite Product Spaces

To start with: Only products of **two σ–fields**!

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Measures on Finite Product Spaces

To start with: Only products of **two σ–fields**!

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3. uniformly σ–finite:
   $\Omega_2 = \bigcup_{n=1}^{\infty} B_n$ where $\mu(\omega_1, B_n) \leqslant k_n$ for all ω_1 and fixed $k_n \in \mathbb{R}$.

Measures on Finite Product Spaces

Theorem (Product Measure Theorem)

Given $(\Omega_1, \mathfrak{F}_1, \mu_1)$, $(\Omega_2, \mathfrak{F}_2)$ *and* $\mu(\omega_1, \cdot)$ *as before.*

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μ *is defined (now on the entire* σ*–field) as follows:*

$$\mu(F) := \int_{\Omega_1} \mu(\omega_1, F(\omega_1))\ \mu_1(d\omega_1), \qquad \text{for all } F \in \mathfrak{F}$$

where $F(\omega_1) := \{\omega_2 \mid (\omega_1, \omega_2) \in F\}$*.*

Lebesgue Integrals on Finite Product Spaces

Theorem (Fubini's Theorem)

Let $f : (\Omega, \mathfrak{F}) \to (\bar{\mathbb{R}}, \mathfrak{B}(\bar{\mathbb{R}}))$. If f is **nonnegative**, *then*

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exists and defines a **Borel measurable** *function. Also*

$$\int_{\Omega} f\ d\mu = \int_{\Omega_1} \Big(\int_{\Omega_2} f(\omega_1, \omega_2)\ \mu(\omega_1, d\omega_2) \Big) \mu_1(d\omega_1).$$

Justification of iterated integration!

Extension to Larger Product Spaces

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1. a measure on $\mathfrak{F}_{j+1}$ and
2. is measurable, i.e. for all fixed $C \in \mathfrak{F}_{j+1}$:

$$\mu(\omega_1, \ldots, \omega_j, C) : \big(\Omega_1 \times \cdots \times \Omega_j, \sigma(\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_j)\big) \to \big(\bar{\mathbb{R}}, \mathfrak{B}(\bar{\mathbb{R}})\big)$$

3. uniformly σ–finite.

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Let $f : (\Omega, \mathfrak{F}) \to (\bar{\mathbb{R}}, \mathfrak{B}(\bar{\mathbb{R}}))$. If $f \geqslant 0$, then

$$\int_{\Omega} f \, d\mu = \int_{\Omega_1} \mu_1(d\omega_1) \int_{\Omega_2} \mu(\omega_1, d\omega_2) \cdots \int_{\Omega_n} f(\omega_1, \ldots, \omega_n) \, \mu(\omega_1, \ldots, \omega_{n-1}, d\omega_n).$$

Measures on Infinite Product Spaces

Definition (Cylinder Set)

Let $(\Omega_j, \mathfrak{F}_j)$ be a measurable space, $j = 1, 2, \ldots$.
Let $\Omega = \times_{j=1}^{\infty} \Omega_j$. If $B^n \subseteq \Omega_1 \times \cdots \times \Omega_n$, define

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- B_n is **measurable** if $B^n \in \sigma(\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n)$.
- B_n is a **rectangle** if $B^n = A_1 \times \cdots \times A_n$ and $A_j \subseteq \Omega_j$;
  B_n is a **measurable rectangle** if $A_j \in \mathfrak{F}_j$.

Measures on Infinite Product Spaces

Ionescu–Tulcea Theorem

Let P_1 be a **probability measure** on $\mathfrak{F}_1$ and **for each** $(\omega_1, \ldots, \omega_j)$, $j \in \mathbb{N}$, assume a measurable probability measure $P(\omega_1, \ldots, \omega_j, \cdot)$ on $\mathfrak{F}_{j+1}$.

Let P_n be defined on $\sigma(\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n)$:

$$P_n(F) = \int_{\Omega_1} P_1(d\omega_1) \int_{\Omega_2} P(\omega_1, d\omega_2) \cdots \int_{\Omega_n} \mathbf{I}_F(\omega_1, \ldots, \omega_n)\, P(\omega_1, \ldots, \omega_{n-1}, d\omega_n).$$

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Intuition: The measure of a cylinder equals the measure of its finite base.

Continuous Time Markov Decision Processes

Definition

A CTMDP is a tuple $\mathcal{C} = (\mathcal{S}, \mathrm{Act}, \mathbf{R}, \mathrm{AP}, \mathrm{L})$ with finite set of states $\mathcal{S}$, labeled according to AP and L. Further

- Act is the set of possible actions and
- $\mathbf{R} : \mathcal{S} \times \mathrm{Act} \times \mathcal{S} \to \mathbb{R}$ is a transition rate matrix, parameterized with actions.

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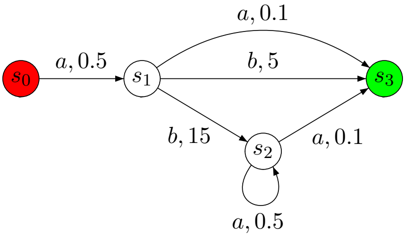
Example

Being in state $s \in \mathcal{S}$,

1. choose enabled action from $\mathrm{Act}(s)$
2. sojourn time in s: $1 - e^{-E(s,a)t}$
3. next state probability: $\frac{\mathbf{R}(s,a,s')}{E(s,a)}$

where $E(s,a) := \sum_{s' \in \mathcal{S}} \mathbf{R}(s,a,s')$.



Paths in a CTMDP

Definition (Paths)

Let $\mathcal{C} = (\mathcal{S}, \mathrm{Act}, \mathbf{R}, \mathrm{AP}, \mathrm{L})$. Finite paths are denoted

$$\pi = s_0 \xrightarrow{a_0, t_0} s_1 \xrightarrow{a_1, t_1} s_2 \xrightarrow{a_2, t_2} \cdots \xrightarrow{a_{n-1}, t_{n-1}} s_n.$$

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Sets of paths are denoted as usual:

$$Paths^n := \mathcal{S} \times (\mathrm{Act} \times \mathbb{R} \times \mathcal{S})^n \qquad \text{and} \qquad Paths^\star := \bigcup_{i=0}^{\infty} Paths^n$$

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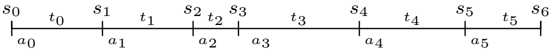
- $\pi@t$ is the state occupied in π at time t.
- $\delta(\pi, n) = t_n$ denotes the time spent in the n–th state.

Paths in a CTMDP

Example



$$\pi @ t := \delta\left(\pi, \min\left\{k \in \mathbb{N} \;\middle|\; \sum_{i=0}^{k} t_i > t\right\}\right) \qquad \delta(\pi, n) := t_n$$

Paths in a CTMDP

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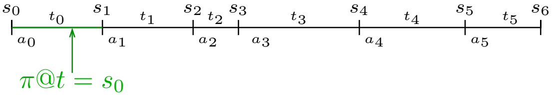

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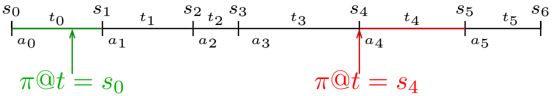

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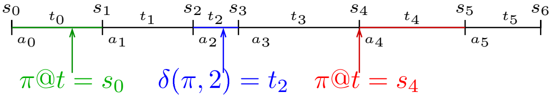

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Paths in a CTMDP

Definition (Infinite Paths)

The set of infinite paths is

$$Paths^{\omega} := \mathcal{S} \times (\mathrm{Act} \times \mathbb{R} \times \mathcal{S})^{\omega}\,.$$

The definitions are extended to $Paths^{\omega}$ if appropriate.

Whatever you like: a, b, c

In CTMDP, the next action is chosen **nondeterministically**.
$\hookrightarrow$ Nondeterminism must be resolved to assign probabilities.

Classes of schedulers

A scheduler resolves the nondeterminism in a $CTMDP$.
According to the information available, distinguish:

1. information about the history:
   stationary markovian, markovian deterministic, history dependent
2. timed or time–abstract

The decision taken can either be

1. deterministic
2. randomized

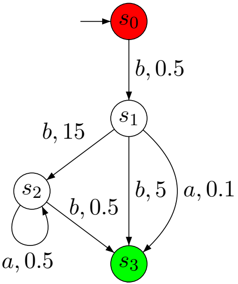
Example: Scheduler Classes

Assume $\mathcal{C} = (\mathcal{S}, \mathrm{Act}, \mathbf{R}, \mathrm{AP}, \mathrm{L})$.

1. Stationary Markovian deterministic scheduler:
   Consider SMD–scheduler that always chooses action b:

$$\mathcal{D} : \mathcal{S} \to \mathrm{Act} : s \mapsto b$$

$\mathcal{C}$ and $\mathcal{D}$ induce a CTMC as follows:



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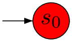

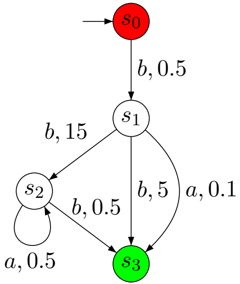

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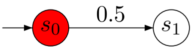

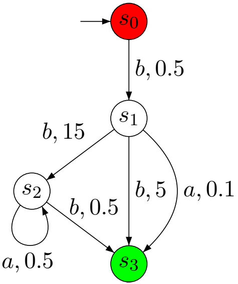

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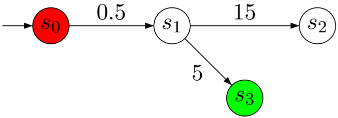

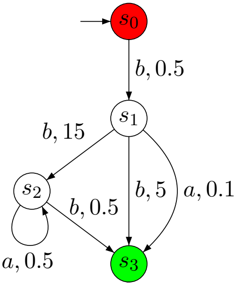

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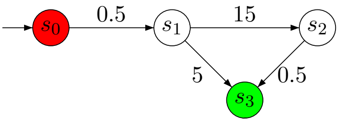

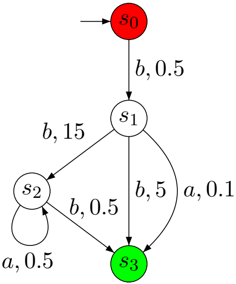

Example: Markovian Randomized Scheduler

Assume $\mathcal{C} = (\mathcal{S}, \mathrm{Act}, \mathbf{R}, \mathrm{AP}, \mathrm{L})$.

2. Markovian randomized scheduler:

$$\mathcal{D} : \mathbb{N} \times \mathcal{S} \to Distr(\mathrm{Act}) : (n, s) \mapsto \begin{cases} \mathbf{I}_b & \text{if } s \in \{s_0, s_1\} \\ \gamma_n & \text{if } s = s_2 \end{cases} \quad \text{where}$$

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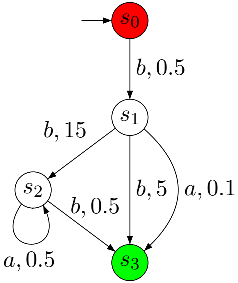

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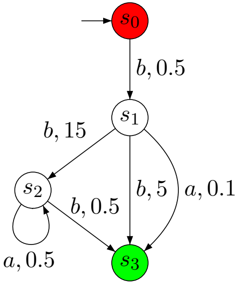

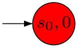

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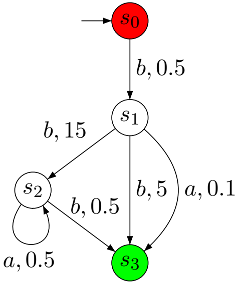

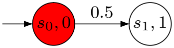

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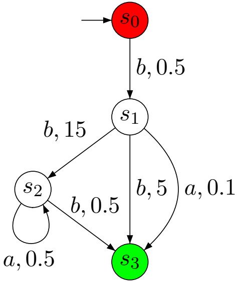

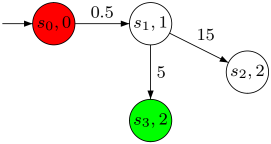

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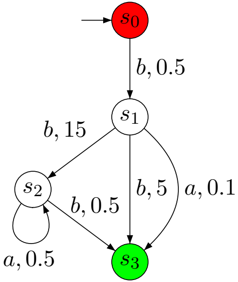

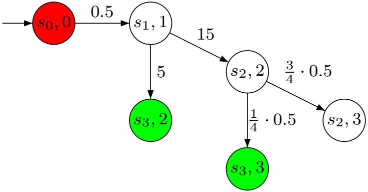

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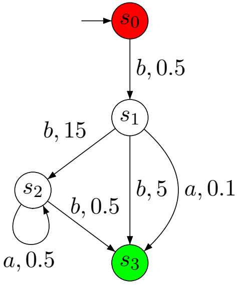

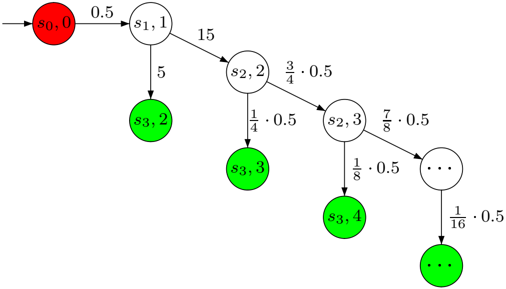

Example: The General Setting

In general: CTMDP $\mathcal{C}$ and scheduler $\mathcal{D}$ do **not** induce a finite (or countable) CTMC!

Example

Consider a timed–history dependent scheduler.
The states of its induced CTMC consist of the set of timed paths which is uncountable.

Semantics: Combined Transitions

Given $\pi \in Paths^\star$, the probability to continue by $\pi\downarrow \xrightarrow{a,t} s$ depends on

- $\mathbf{R}(\pi\downarrow, a, s)$, the exponential distribution of CTMDP and
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Definition (Combined Transition)

Let $\Omega = \mathrm{Act} \times \mathbb{R} \times \mathcal{S}$. Then $(a, t, s) \in \Omega$ is a **combined transition**.

- $\mathfrak{F}_{\mathrm{Act}} \times \mathfrak{B}(\mathbb{R}) \times \mathfrak{F}_{\mathcal{S}}$ is the class of **measurable rectangles** and
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Example

$\mathfrak{F}$ is the class of measurable sets of combined transitions;
$M \in \mathfrak{F}$ is a set of combined transitions.

Semantics: From Paths to Rectangles

Finite Measurable Path Rectangles

A set of path of length n, represented as a Cartesian product

$$S_0 \times \underbrace{A_0 \times I_0 \times S_1}_{M_0} \times \cdots \times \underbrace{A_{n-1} \times I_{n-1} \times S_n}_{M_{n-1}}$$

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Lemma

The class of finite disjoint unions of measurable rectangles is a field.

Semantics: The Product σ–Field

Finite Product σ–Field over Measurable Path Rectangles

The smallest σ–field generated by **measurable path rectangles**:

$$\mathfrak{F}_{Paths^n} := \sigma\Big(\mathfrak{F}_{\mathcal{S}} \times \mathfrak{F}^n\Big) \qquad \text{for } n \geqslant 0.$$

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Let $\mathcal{S} = \{s_0, s_1\}$, $\mathrm{Act} = \{a, b\}$.

- $\{s_0\} \times \{a, b\} \times (0, 0.2] \cup [1.2, 2] \times \{s_0, s_1\}$ is a measurable **rectangle**.
  Elements: $s_0 \xrightarrow{a,0.1} s_0$, $s_0 \xrightarrow{a,0.1001} s_0$, $s_0 \xrightarrow{b,\sqrt{2}} s_1$, etc.

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Semantics: Cylinders and Infinite Paths I

Cylinder Set Construction

Let $C^n \subseteq Paths^n$ be a set of finite paths. Its induced **cylinder** is

$$C_n := \{\pi \in Paths^\omega \mid \pi[0..n] \in C^n\}$$

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- If $m < n$ and $C^m = C^n \times \Omega^{n-m}$, then $C_m = C_n$.

Semantics: Cylinders and Infinite Paths II

Definition (σ–Field generated by Measurable Cylinders)

The minimal σ–field generated by measurable cylinders is defined by

$$\mathfrak{F}_{Paths^\omega} := \sigma\Big(\mathfrak{F}_{\mathcal{S}} \times \mathfrak{F}^\infty\Big)$$

or equivalently

$$\mathfrak{F}_{Paths^\omega} := \sigma\Big(\bigcup_{i=0}^{\infty} \{C_n \mid C^n \in \mathfrak{F}_{Paths^n}\}\Big).$$

Finally: $(Paths^\omega, \mathfrak{F}_{Paths^\omega})$ is our measurable space.

Semantics: Combined Transition Probability

Product Measure on Combined Transitions

For history $\pi \in Paths^{\star}$, three types of measure spaces are involved:

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Definition (A Measure on Subsets of Ω)

Let $\pi \in Paths^{\star}$. Then

$$\mu_{\mathcal{D}}(\pi, \cdot) : \mathfrak{F} \to [0,1] :$$

$$M \mapsto \int_{\mathrm{Act}} \mathcal{D}(\pi, da) \int_{\mathbb{R}_{\geqslant 0}} \eta_a(dt) \int_{\mathcal{S}} \overbrace{\mathbf{I}_M(a,t,s)}^{\text{indicator}} \mathbf{P}(\pi\downarrow, a, ds).$$

Semantics: Combined Transition Probability

Example (Probability Measure of Rectangles)

Let $\pi \in Paths^\star$. Let $A \times I \times S' \in \mathfrak{F}$, I an interval. Then

$$\mu_{\mathcal{D}}(\pi, A \times I \times S') = \sum_{a \in A} \mathcal{D}(\pi, \{a\}) \cdot \mathbf{P}(\pi\downarrow, a, S') \cdot \int_I E(\pi\downarrow, a) \cdot e^{-E(\pi\downarrow, a)t} dt$$

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Semantics: Combined Transition Probability

Lemma (Measurability of $\mu_{\mathcal{D}}$)

For fixed $M \in \mathfrak{F}$ and finite path–length n:

$$\mu_{\mathcal{D}}(\cdot, M) : (Paths^n, \mathfrak{F}_{Paths^n}) \to (\mathbb{R}, \mathfrak{B}(\mathbb{R}))$$

Measurability **necessary** for Lebesgue–integration.

Semantics: Probability Measures on $\mathfrak{F}_{Paths^\star}$

Definition (Probability Measures on $\mathfrak{F}_{Paths^\star}$)

Let $(\mathcal{S}, \mathrm{Act}, \mathbf{R}, \mathrm{AP}, \mathrm{L})$ be a CTMDP, α an initial distribution and $\mathcal{D}$ a THR scheduler.

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$$Pr^0_{\alpha,\mathcal{D}} : \mathfrak{F}_\mathcal{S} \to [0,1] : S \mapsto \sum_{s \in S} \alpha(s)$$

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$$\Pi \mapsto \int_{Paths^{n-1}} Pr^{n-1}_{\alpha,\mathcal{D}}(d\pi) \int_\Omega \overbrace{\mathbf{I}_\Pi(\pi \circ m)}^{\text{indicator}} \; \mu_\mathcal{D}(\pi, dm).$$

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Intuition: Probability of Rectangles

Example (Probability of Rectangles)

For measurable rectangle $\Pi \times M \in \mathfrak{F}_{Paths^n}$, we obtain

$$Pr^n_{\alpha,\mathcal{D}}(\Pi \times M) = \int_{\Pi} \mu_{\mathcal{D}}(\pi, M)\ Pr^{n-1}_{\alpha,\mathcal{D}}(d\pi)$$

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$$Pr^n_{\alpha,\mathcal{D}}(\Pi \times M) = \int_{\Pi} \mu_{\mathcal{D}}(\pi, M)\ Pr^{n-1}_{\alpha,\mathcal{D}}(d\pi)$$

where

$$\mu_{\mathcal{D}}(\pi, M) = \int_{\mathrm{Act}} \mathcal{D}(\pi, da) \int_{\mathbb{R}_{\geqslant 0}} \eta_a(dt) \int_{\mathcal{S}} \underbrace{\mathbf{I}_M(a, t, s)}_{\text{indicator}}\ \mathbf{P}(\pi\downarrow, a, ds).$$

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Intuition: Probability of Rectangles

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Intuition:

- $\Pi \in \mathfrak{F}_{Paths^{n-1}}$ is a measurable set of paths,
- $M \in \mathfrak{F}$ is a set of combined transitions (e.g. $M = A \times I \times S'$).

Semantics: Probability of Measurable Cylinders

A Probability Measure on $\mathfrak{F}_{Paths^\omega}$

Any measurable cylinder C can be represented as

$$C = \{\pi \in Paths^\omega \mid \pi[0..n] \in C^n\} \quad \text{for some } n \geqslant 0 \text{ and } C^n \in \mathfrak{F}_{Paths^n}.$$

Define the probability measure on measurable cylinders:

$$Pr^\omega_{\alpha,\mathcal{D}} : \mathfrak{F}_{Paths^\omega} \to [0,1] : C_n \mapsto Pr^n_{\alpha,\mathcal{D}}(C^n)$$

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Theorem (Ionescu–Tulcea)

$Pr^\omega_{\alpha,\mathcal{D}}$ *is well–defined and unique.*

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Finally: $\left(Paths^\omega, \mathfrak{F}_{Paths^\omega}, Pr^\omega_{\alpha,\mathcal{D}}\right)$ is the desired probability space.

The Logic $nCSL$

*n*ondeterministic *C*ontinuous *S*tochastic *L*ogic ($nCSL$).

Zeno Behaviour

Definition (Zeno Path)

Let $\mathcal{C} = (\mathcal{S}, \mathrm{Act}, \mathbf{R}, \mathrm{AP}, \mathrm{L})$ be a CTMDP and

$$\pi = s_0 \xrightarrow{a_0, t_0} s_1 \xrightarrow{a_1, t_1} s_2 \xrightarrow{a_2, t_2} \cdots .$$

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Lemma (Converging Paths Lemma)

The probability measure of the set of converging paths is zero.

Example (What is it good for?)

For any $\pi \in Paths^{\omega}$ and $t \in \mathbb{R}_{\geqslant 0}$, $\pi@t$ is well–defined.

Syntax of $nCSL$

Two kinds of property specifications:

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Example (Long Run Average Behaviour)

Average time spent in a blocking state.

Syntax of $nCSL$

Definition ($nCSL$ Formulae)

For $a \in \mathrm{AP}$, $p \in [0,1]$ and $\sqsubseteq \in \{<, \leqslant, \geqslant, >\}$, $nCSL$ **state–formulas** are built according to the following context–free grammar:

$$\Phi ::= \ a \mid \neg\Phi \mid \Phi \wedge \Phi \mid \exists^{\sqsubseteq p}\varphi \mid \mathsf{L}^{\sqsubseteq p}\Phi$$

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For $I \subseteq \mathbb{R}$ a nonempty interval, $nCSL$ **path–formulas** are defined by

$$\varphi ::= \; \mathsf{X}^I\Phi \mid \Phi\mathsf{U}^I\Phi$$

Semantics of $nCSL$ I

Definition (Semantics of State Formulae)

Let $\mathcal{C} = (\mathcal{S}, \mathrm{Act}, \mathbf{R}, \mathrm{AP}, \mathrm{L})$. Define

$$s \models a \iff a \in \mathrm{L}(s)$$
$$s \models \neg\Phi \iff \text{ not } s \models \Phi$$
$$s \models \Phi \wedge \Psi \iff s \models \Phi \text{ and } s \models \Psi$$

Semantics of $nCSL$: Path Formulae

Definition (Semantics of Path Formulae)

For time–interval $I \subseteq \mathbb{R}$ and state formulas Φ and Ψ, define:

$$\pi \models \mathsf{X}^I \Phi \iff \pi[1] \models \Phi \wedge \delta(\pi, 0) \in I$$

$$\pi \models \Phi \mathsf{U}^I \Psi \iff \exists t \in I.\ \big(\pi@t \models \Psi \wedge \big(\forall t' \in [0, t).\ \pi@t' \models \Phi\big)\big).$$

Semantics of $nCSL$: Transient State Measures

Definition (Transient State Formulae)

For probability bound $p \in [0, 1]$, comparison operator $\sqsubseteq$ and path formula φ, the transient state semantics is given by

$$s \models \exists^{\sqsubseteq p} \varphi \Longleftrightarrow \exists D \in THR.\ Pr^{\omega}_{\alpha,\mathcal{D}_s} \{\pi \in Paths^{\omega} \mid \pi \models \varphi\} \sqsubseteq p$$

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Lemma (Measurability of Satisfying Paths)

For arbitrary path formula φ:

$$\{\pi \in Paths^{\omega} \mid \pi \models \varphi\} \in \mathfrak{F}_{Paths^{\omega}}.$$

Semantics of $nCSL$: Long Run Average Behaviour I

Preliminaries

For CTMDP $\mathcal{C} = (\mathcal{S}, \mathrm{Act}, \mathbf{R}, \mathrm{AP}, \mathrm{L})$, state s and state–formula Φ:

What is the average amount of time spent in Φ–states?

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Definition (state indicator)

Let $S \subseteq \mathcal{S}$ and $t \in \mathbb{R}_{\geqslant 0}$. Then

$$h_{S,t} : Paths^{\omega} \rightarrow \{0,1\} : \pi \mapsto \begin{cases} 1 & \text{if } \pi@t \in S \\ 0 & \text{otherwise} \end{cases}$$

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Intuition: Does π occupy a state from S at time–point t?

Lemma

The function $h_{S,t}$ is measurable relative to $(Paths^{\omega}, \mathfrak{F}_{Paths^{\omega}})$.

Semantics of $nCSL$: Long Run Average Behaviour II

Deduction of Long Run Average Behaviour

The fraction of time spent in S–states on path $\pi \in Paths^{\omega}$:

$$g_{S,t} : Paths^{\omega} \rightarrow [0,1] : \pi \mapsto \frac{1}{t} \int_0^t h_{S,t'}(\pi)\ dt'.$$

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Lemma

$g_{S,t}$ *is* **measurable** *in* $(Paths^{\omega}, \mathfrak{F}_{Paths^{\omega}})$ *and a* **random variable**.

Definition (Expectation)

Take the expectation $g_{S,t}$ over $\pi \in Paths^{\omega}$:

$$E(g_{S,t}) = \int_{Paths^{\omega}} g_{S,t}(\pi)\ Pr^{\omega}_{\alpha,\mathcal{D}}(d\pi).$$

Semantics of $nCSL$: Long Run Average Behaviour III

Definition (Long Run Average Formulae)

For fraction $p \in [0,1]$, comparison operator $\sqsubseteq$ and state formula Φ, the long–run average semantics is defined by:

$$s \models \mathsf{L}^{\sqsubseteq p}\Phi \iff \forall D \in THR. \lim_{t \to \infty} \underbrace{\int_{Paths^\omega} \left(\frac{1}{t} \int_0^t h_{Sat(\Phi),t'}(\pi) dt' \right) dPr^\omega_{\alpha_s,\mathcal{D}}}_{\text{expectation!}} \sqsubseteq p$$

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Example

For CTMDP $\mathcal{C}$ with initial state s where $r \in \mathrm{AP}$ labels all **reactive states**. The property

"99% of the time, the system directly reacts on input"

can be checked by the following $nCSL$–formula:

$$s \models \mathsf{L}^{>0.99} r$$

Ongoing Work

1. Complete measurability issues in $nCSL$–semantics
   Measurability of U^I subformulas.
2. Are all $nCSL$–formulas preserved under strong bisimulation?
   Provide the proof.
3. Which $nCSL$–restrictions are preserved under strong simulation?
   Define strong simulation on CTMDP, find appropriate restriction of $nCSL$

Thank you for your attention!