Model Checking Nondeterministic and Randomly Timed Systems

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Oberseminar on January 25, 2010, RWTH Aachen University

The challenge

Software in safety critical systems becomes more and more complex.

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Erroneous behavior of complex hardware & software systems

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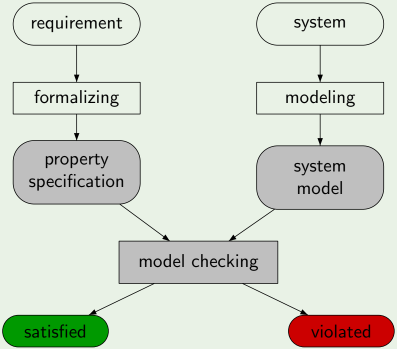
Erroneous behavior of complex hardware & software systems

Formal methods in computer science

The model checking approach



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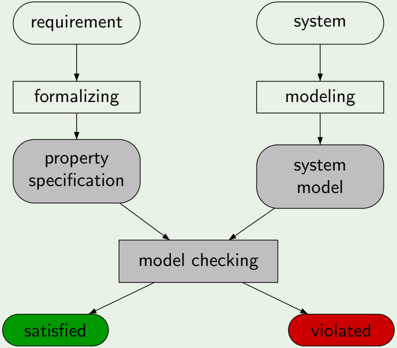

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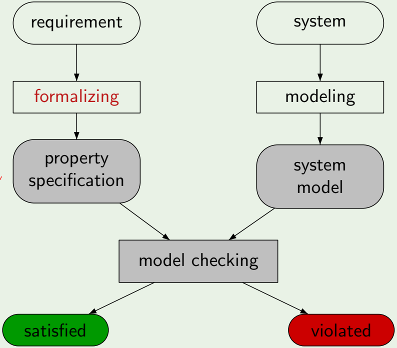
The model checking approach

$$\Phi = \forall \Box \neg \textit{collision}$$



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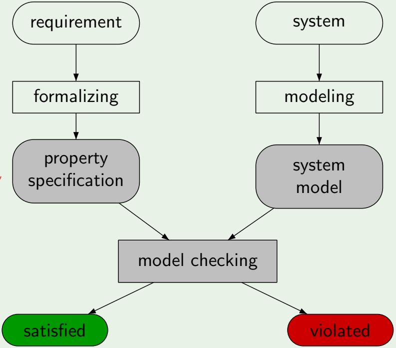

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Formal methods in computer science

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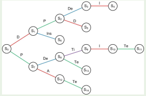
requirement → formalizing → property specification

system → modeling → system model

property specification, system model → model checking

model checking → satisfied

model checking → violated



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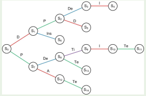

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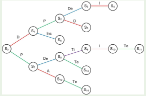

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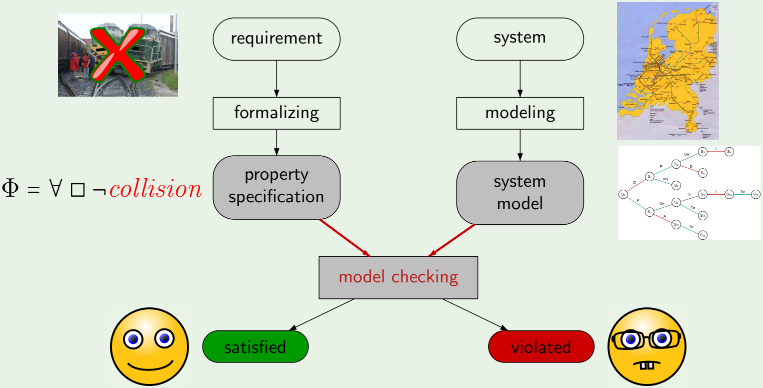

Model checking: Does a system model satisfy its specification?

Quantitative system analysis

Classical model checking

Model checking yields **YES** or **NO**.
But: Absolute correctness unrealistic:

- Systems are subject to random phenomena
- Environment behaves randomly
- Imprecisions in the model

Quantitative model checking

Extend models with probabilities to describe real-world systems
$\Rightarrow$ Performance & dependability evaluation

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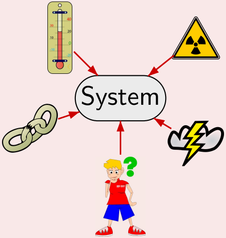

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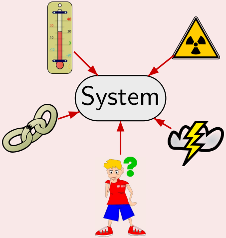


Quantitative model checking

Extend models with probabilities to describe real-world systems
$\Rightarrow$ **Performance & dependability evaluation**

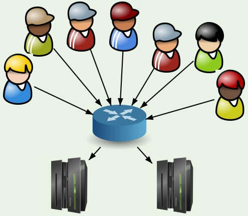
Outline of the talk

1. Introduction
2. Continuous-time Markov decision processes (CTMDPs)
   - Motivation
   - Preliminaries
   - Resolving nondeterministic choices
3. Time-bounded reachability analysis in CTMDPs
   - The approximation algorithm
   - Solving the sJSP
4. Further results in the thesis
   - Model checking interactive Markov chains
   - Model checking generalized stochastic Petri nets
5. Conclusion

Motivation: The stochastic job scheduling problem

Application: Load balancing of a bank's website

1. Customer request ≡ job.
2. Distribute request to multiple servers.
3. Classify jobs according to exp. duration:
   - Online Banking: long job
   - Serving ticker: short job



Clever way to distribute jobs to servers?

The problem statement [Bruno,Downey,Frederickson '81]

- Four jobs $\{1, 2, 3, 4\}$
- Expected duration of job k is $\frac{1}{\lambda_k}$ time units.
- Two identical processors.

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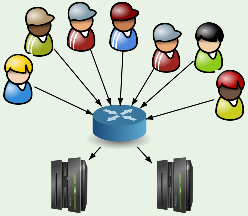

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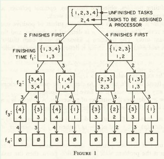

Figure 1

Motivation: The stochastic job scheduling problem

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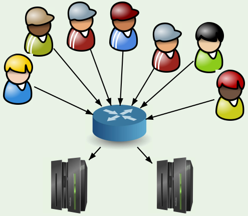

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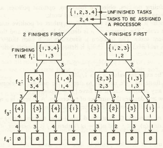

FIGURE 1

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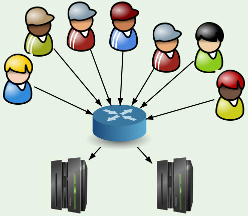

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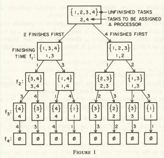

Today: Compute maximum probability to finish all jobs within time z!

Formalizing the stochastic job scheduling problem

Preemptive scheduling of 4 jobs onto 2 processors

$$\longrightarrow \boxed{1,2,3,4}$$

In this talk:

- Compute the maximum probability to finish all jobs before time ...
- Synthesize optimal schedule to achieve this probability

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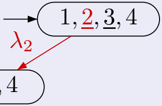

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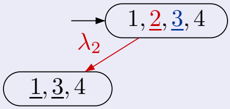

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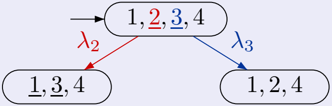

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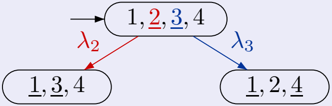

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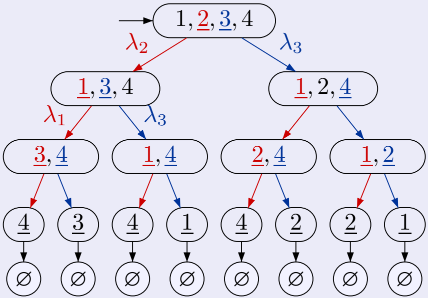

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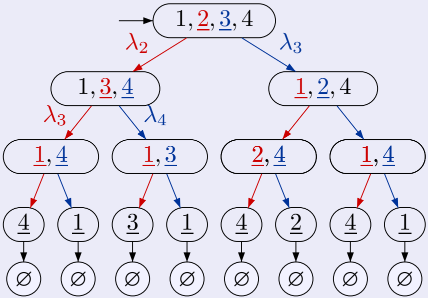

Many schedules possible!

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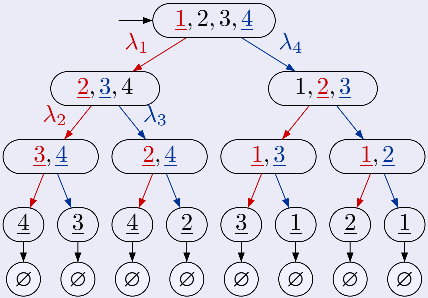

Many schedules possible!

In this talk:

- Compute the maximum probability to finish all jobs before time t.
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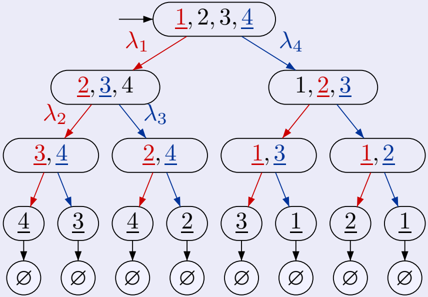
Formalizing the stochastic job scheduling problem

Preemptive scheduling of 4 jobs onto 2 processors



Many schedules possible!

In this talk:

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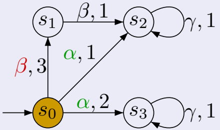
Continuous-time Markov decision processes (CTMDPs)

Random timing and nondeterminism in CTMDPs



Initial state: s_0
Actions: $Act(s_0) = \{\alpha, \beta\}$
Choice is nondeterministic!
Transition rates: $\mathbf{R}(s_0, \alpha, s_3) = 2$

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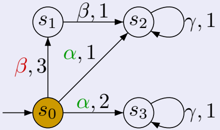

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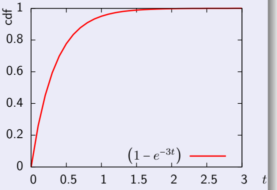
If action β is chosen: Only one transition available

$s_0 \xrightarrow{\beta, 3} s_1$ executes after $X_1 \sim Exp(3)$ time units.

Probability to move before t time units:

$$P(X_1 \leq t) = \int_0^t 3 \cdot e^{-3x}\, dx = \left(1 - e^{-3t}\right)$$



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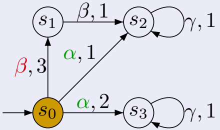

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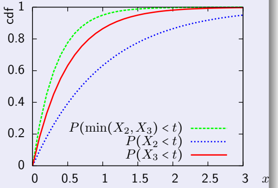
If action α is chosen: Race condition

$s_0 \xrightarrow{\alpha, 1} s_2$ executes after $X_2 \sim Exp(1)$ time units.
$s_0 \xrightarrow{\alpha, 2} s_3$ executes after $X_3 \sim Exp(2)$ time units.



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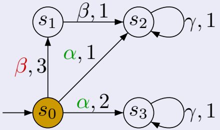

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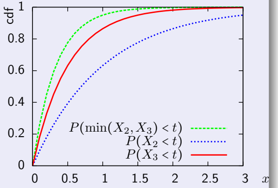


The transition that executes first, wins:

1. Time spent in s_0: $\min(X_2, X_3) \sim Exp(1 + 2)$
   Exit rate: $E(s, \alpha) = \sum_{s' \in \mathcal{S}} \mathbf{R}(s, \alpha, s') = 1 + 2$
2. Prob. to move to $s_2 = P(X_2 < X_3)$.

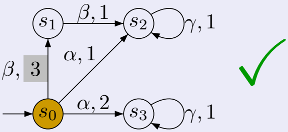
The subclass of locally uniform CTMDPs

Restriction to local uniformity

A CTMDP is locally uniform iff

$$\forall s \in \mathcal{S}.\ \forall \alpha, \beta \in Act(s). \quad E(s,\alpha) = E(s,\beta).$$

Exit rate $E(s)$ independent of action!



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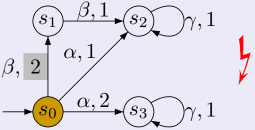

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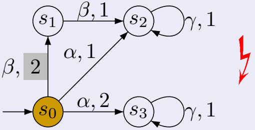


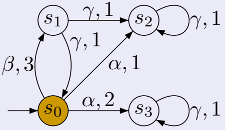
Why this restriction?

Sojourn time in a state does not depend on action!

Resolving nondeterministic choices

Resolving nondeterminism by schedulers:

$\pi = s_0$



Time & history dependent schedulers

A scheduler is a measurable mapping

$$D : Paths^{\star} \times \mathbb{R}_{\geq 0} \to Distr(Act)$$

such that $D(\pi, t)(\alpha) > 0 \Rightarrow \alpha \in Act(last(\pi))$.

The probability measure

Scheduler D & initial state s induce unique probability measure

Resolving nondeterministic choices

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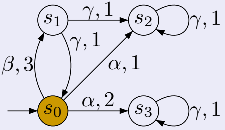
wait in s_0 for $X_0 \sim Exp(3)$ time units

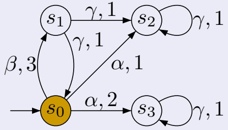


Resolving nondeterministic choices

Resolving nondeterminism by schedulers:

$$\pi = s_0 \xrightarrow{t_0} ?$$

upon leaving s_0: nondeterministic choice!

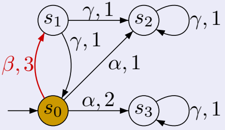


Resolving nondeterministic choices

Resolving nondeterminism by schedulers:

$$\pi = s_0 \xrightarrow{\beta, t_0} ?$$

$$D\left(s_0, t_0\right) = \beta$$



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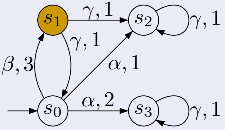
$$\pi = s_0 \xrightarrow{\beta, t_0} s_1$$

wait in s_1 for $X_1 \sim Exp(2)$ time units

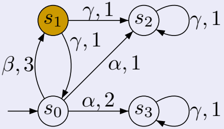


Resolving nondeterministic choices

Resolving nondeterminism by schedulers:

$$\pi = s_0 \xrightarrow{\beta, t_0} s_1 \xrightarrow{t_1} ?$$

upon leaving s_1: only γ available.

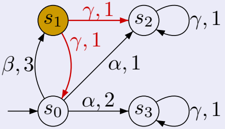


Resolving nondeterministic choices

Resolving nondeterminism by schedulers:

$$\pi = s_0 \xrightarrow{\beta, t_0} s_1 \xrightarrow{\gamma, t_1} ?$$

Race: $\frac{\mathbf{R}(s_1,\gamma,s_2)}{E(s_1)} = \frac{1}{2}$ chance to move to s_0



Resolving nondeterministic choices

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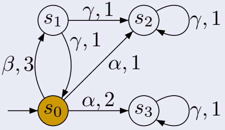
wait in s_0 for $X_0 \sim Exp(3)$ time units



Time & history dependent schedulers

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$$D : Paths^* \times \mathbb{R}_{\geq 0} \to Distr(Act)$$

such that $D(\pi, t)(\alpha) > 0 \Rightarrow \alpha \in Act(last(\pi))$.

The probability measure

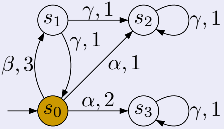
Scheduler D & initial state s induce unique probability measure $Pr^{\omega}_{D,s}$

Resolving nondeterministic choices

Resolving nondeterminism by schedulers:

$$\pi = s_0 \xrightarrow{\beta, t_0} s_1 \xrightarrow{\gamma, t_1} s_0 \xrightarrow{t_2} ?$$

nondeterministic choice between α and β



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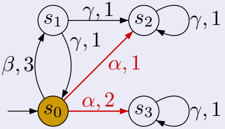
$$D\left(s_0 \xrightarrow{\beta, t_0} s_1 \xrightarrow{\gamma, t_1} s_0, t_2\right) = \alpha$$



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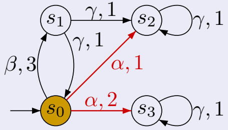
Race: $\frac{\mathbf{R}(s_0,\alpha,s_2)}{E(s_0)} = \frac{1}{3}$ chance to move to s_2



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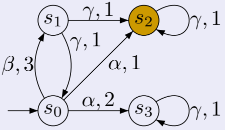

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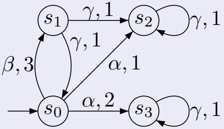

Time & history dependent schedulers [Neuhäußer,Stoelinga,Katoen '09]

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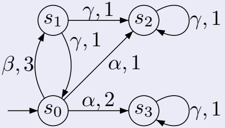

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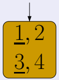
Modeling the sJSP as a CTMDP

Finishing all jobs within z time units:



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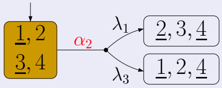

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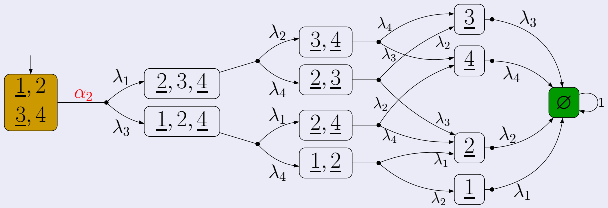

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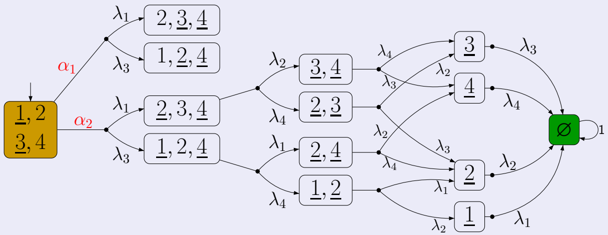

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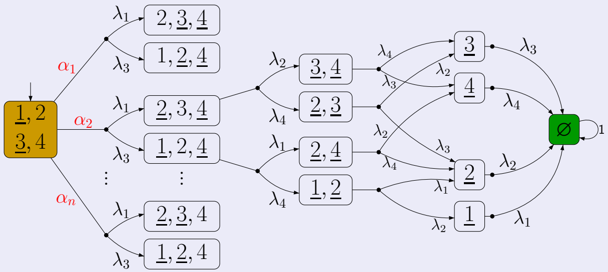

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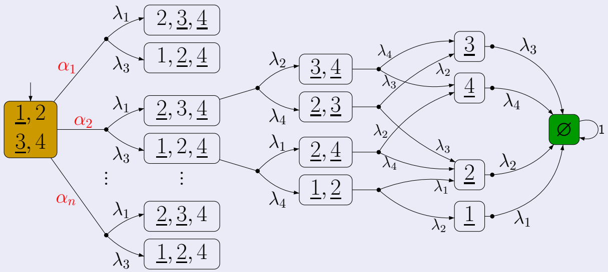

All scheduling strategies represented in the CTMDP.

$\alpha_1 : (1 \mapsto \{3,4\}, 3 \mapsto \{2,4\})$ $\alpha_2 : (1 \mapsto \{2,4\}, 3 \mapsto \{1,4\})$ ...

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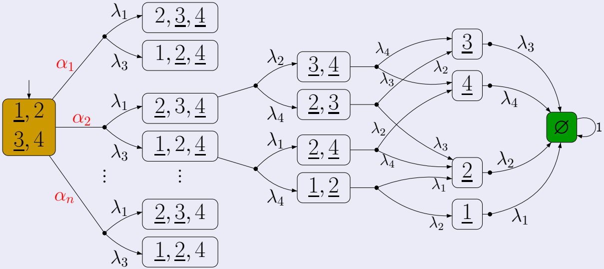

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Properties:

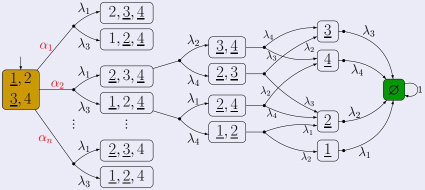
1. CTMDP combines nondeterministic choices and stochastic timing.
2. The CTMDP model is locally uniform.

Outline of the talk

Time-bounded reachability in the sJSP

Time-bounded reachability probabilities

- CTMDP model $\mathcal{C}$.
- Initial state: $s \in \mathcal{S}$
- Goal states: $G \subseteq \mathcal{S}$
- Time-bound: $z \in \mathbb{R}_{\geq 0}$



The time-bounded reachability event:

$$\Diamond^{[0,z]} G = \left\{ \pi \in Paths^{\omega} \mid \exists t \in [0,z].\ \pi@t \in G \right\}$$

Maximum time-bounded reachability probability

$$p^{\mathcal{C}}_{max}(s,z) = \sup_{D} Pr^{\omega}_{s,D}\left(\Diamond^{[0,z]} G\right)$$

Computing the maximum time-bounded reachability probability

How to compute $p^{\mathcal{C}}_{max}$?

Idea: Characterize $p^{\mathcal{C}}_{max}$ as a fixed-point!

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A higher operator for maximum time-bounded reachability

Define $\Omega : (\mathcal{S} \times \mathbb{R}_{\geq 0} \to [0,1]) \to (\mathcal{S} \times \mathbb{R}_{\geq 0} \to [0,1])$ on measurable functions:

- If $s \in G$ then $\Omega(F)(s,z) = 1$.
- If $s \notin G$ then

$$\Omega(F)(s,z) = \int_0^z E(s)e^{-E(s)t} \cdot \max_{\alpha \in Act} \sum_{s' \in S} \mathbf{P}(s,\alpha,s') \cdot F(s', z-t)\, dt.$$

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Fixed point characterization

The function $p^{\mathcal{C}}_{max}(s,z)$ is the least fixed point of Ω.

Applying the fixed point characterization directly

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If $s \in G$: $\Omega(F)(s,z) = 1$. Otherwise:

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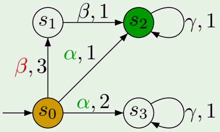
Fixed-point computation: $\forall s \in \mathcal{S}.\ F_0(s,z) = 0.$



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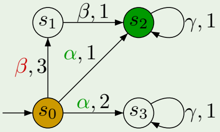

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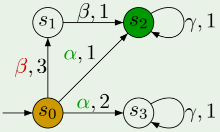

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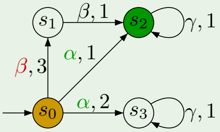

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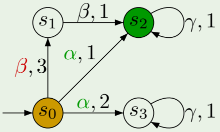

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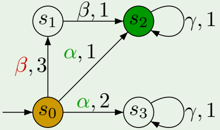

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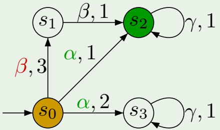

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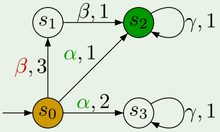

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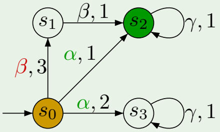

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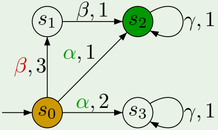

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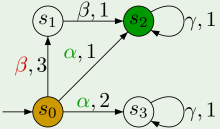

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If $s \in G$: $\Omega(F)(s,z) = 1$. Otherwise:

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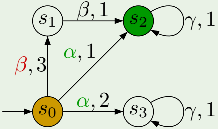

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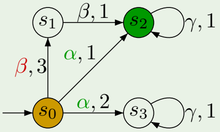

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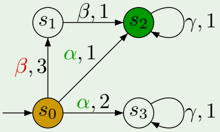

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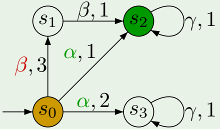

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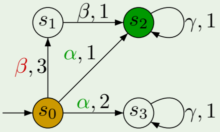

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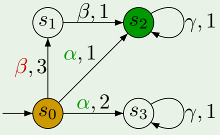

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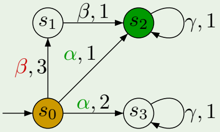

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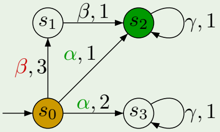

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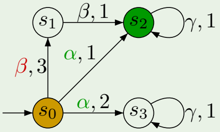

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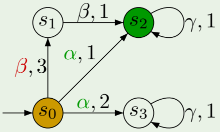

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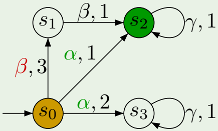

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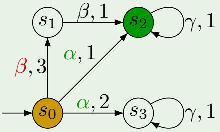

Result: $F_3 = \Omega(F_3) \Rightarrow F_3$ is least fixed-point.
For $z = 1$: $p^{\mathcal{C}}_{max}(s_0, 1) = 1 + \frac{19}{24}e^{-3} - \frac{3}{2}e^{-1} \simeq 0.48759.$

What is achieved so far:

Analytical solution

Allows to compute $p^{\mathcal{C}}_{max}(s, z)$ for small problem instances.

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Instead:

Use the discretization technique that comes next!

A discretization that computes $p_{max}(s, z)$

Reduce $p^{\mathcal{C}}_{max}$ to step-bounded reachability $p^{\mathcal{C}_\tau}_{max}$ in MDPs.

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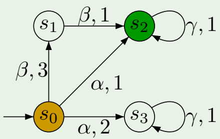
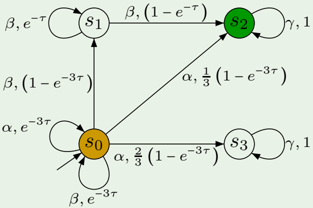
Continuous-time vs. discrete-time Markov decision processes

Continuous-time MDP $\mathcal{C}$



Exponential distributions

Discrete-time MDP $\mathcal{C}_\tau$



Discrete probability distributions

Reachability within time z $\equiv$ **Reachability in $\frac{z}{\tau}$ steps!**

$p^{\mathcal{C}}_{max}(s, z)$ $\qquad$ $p^{\mathcal{C}_\tau}_{max}(s, \frac{z}{\tau})$

Discretization I

Recall the fixed-point characterization:

The function $p^{\mathcal{C}}_{max}(s, z)$ is the least fixed point of Ω: If $s \notin G$, then

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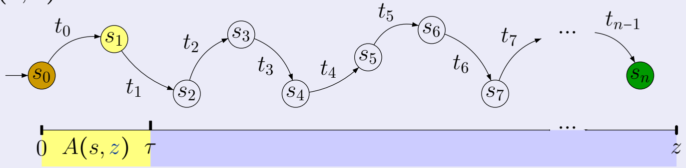
Relation to $p^{\mathcal{C}}_{max}(s,z)$: $p^{\mathcal{C}}_{max}(s,z) = A(s,z) + B(s,z)$

Discretization II

Intuition behind $A(s,z)$ and $B(s,z)$

- $A(s,z)$ = Prob. to reach G within time z with ≥ 1 **transitions in** $[0,\tau]$.



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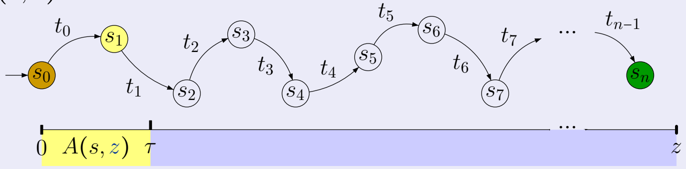
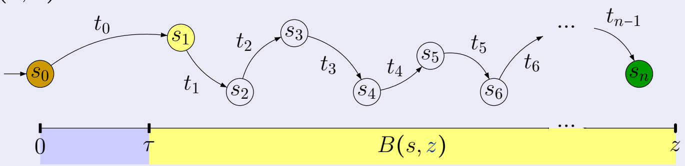


- $B(s, z)$ = Prob. to reach G within time z with **no transition in** $[0, \tau]$.



A step-wise approximation of $p^{\mathcal{C}}_{max}(s, z)$

A single step in the discretized MDP

CTMDP $\mathcal{C}$ and step duration $\tau < z$ induce the discretized MDP $\mathcal{C}_\tau$:

$$\mathbf{P}_\tau(s, \alpha, s') = \begin{cases} \left(1 - e^{-E(s)\tau}\right) \cdot \mathbf{P}(s, \alpha, s') & \text{if } s \neq s' \\ \left(1 - e^{-E(s)\tau}\right) \cdot \mathbf{P}(s, \alpha, s) + e^{-\lambda(s)\tau} & \text{if } s = s'. \end{cases}$$

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Theorem (Correctness of our reduction)

Let $\mathcal{C}$ be a CTMDP, G a set of goal states and z a time bound. Choose some $k \in \mathbb{N}_{>0}$ and set $\tau = \frac{z}{k}$. Then

$$p^{\mathcal{C}_\tau}_{max}(s,k) \leq p^{\mathcal{C}}_{max}(s,z) \leq p^{\mathcal{C}_\tau}_{max}(s,k) + \frac{(\lambda z)^2}{2k}.$$

1. *$p^{\mathcal{C}_\tau}_{max}(s,k)$ is the probability to reach G in at most k steps in $\mathcal{C}_\tau$,*
2. *$\lambda = max_{s\in\mathcal{S}} E(s)$ is the maximum exit rate in $\mathcal{C}$ and*
3. *k is the number of discretization steps.*

Solving the step-bounded reachability problem in $\mathcal{C}_\tau$

Value iteration for discrete-time MDPs [Bellman '57]

Let $G \subseteq \mathcal{S}$ be a set of goal states and $\vec{v}_n \in [0,1]^{|\mathcal{S}|}$ such that

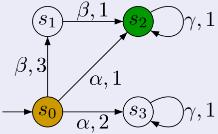
$$\vec{v}_0(s) = \begin{cases} 1 & \text{if } s \in G \\ 0 & \text{if } s \notin G \end{cases} \qquad \vec{v}_{n+1}(s) = \begin{cases} 1 & \text{if } s \in G \\ \max\limits_{\alpha \in Act} \sum\limits_{s' \in \mathcal{S}} \mathbf{P}(s,\alpha,s') \cdot v_n(s') & \text{if } s \notin G \end{cases}$$

Then $p_{max}^{\mathcal{C}_\tau}(s,k) = \vec{v}_k(s)$.

Summarizing our time-bounded reachability analysis

Input

1. locally uniform CTMDP
2. Goal states: $G = \{s_2\}$
3. Time bound: $z = 1$
4. Maximum allowed error: $\varepsilon = 10^{-3}$



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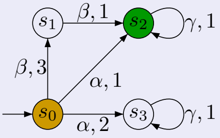

Compute the number of discretization steps k

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Summarizing our time-bounded reachability analysis

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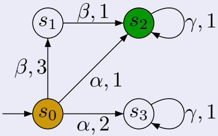

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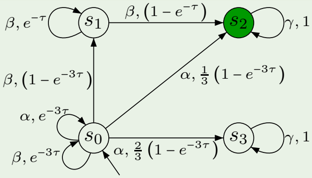

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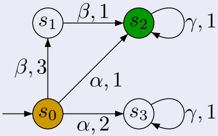

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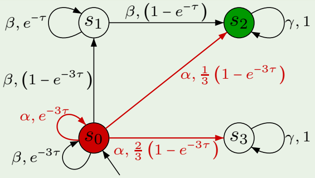

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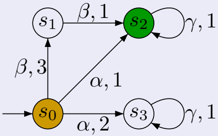

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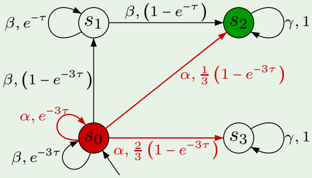

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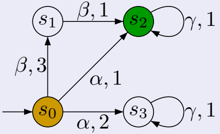

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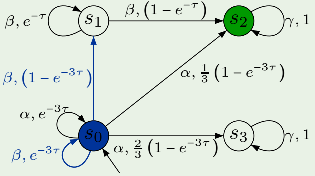

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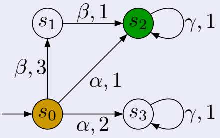

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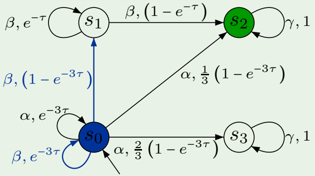

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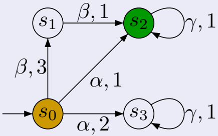

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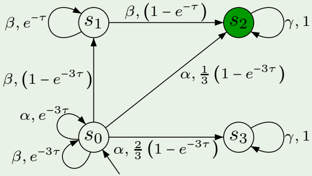

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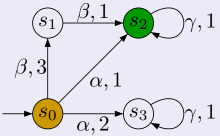

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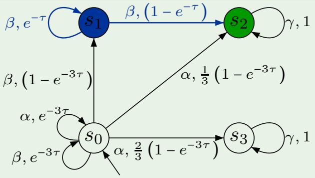

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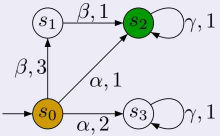

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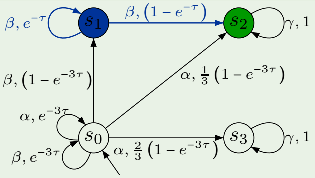

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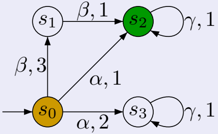

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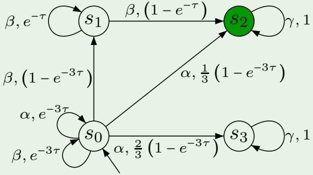

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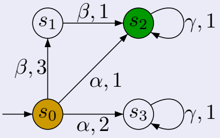

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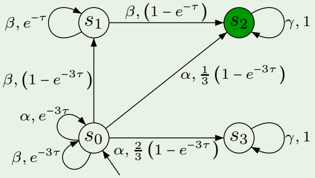

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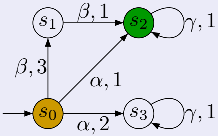

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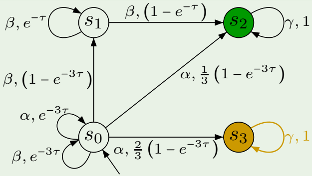

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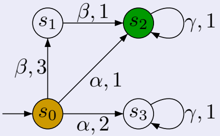

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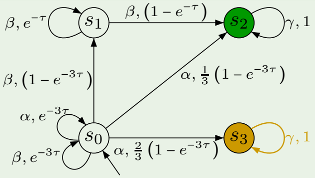

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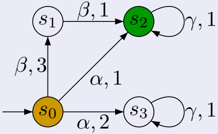

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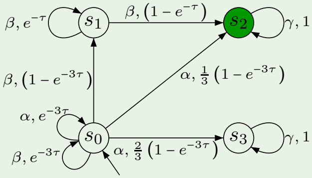

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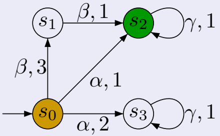

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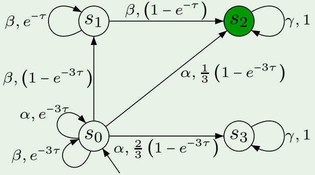
Result: $p_{max}^{\mathcal{C}_\tau}(s_0, 4500) = \vec{v}_{4500}(s_0) \simeq 0.487$



Complexity of the discretization approach

Complexity

For CTMDP $\mathcal{C}$, time bound z and error bound ε:

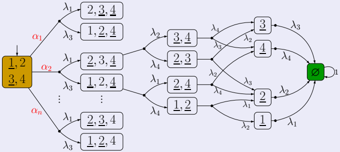
- Number of iteration steps: $\mathcal{O}\big((\lambda z)^2/\varepsilon\big)$.
- Each value iteration step: linear in the size of $\mathcal{C}$ (transitions + states)

Overall complexity: $\mathcal{O}\big(|\mathcal{C}| \cdot (\lambda z)^2/\varepsilon\big)$.

Outlook: Computing optimal solutions for the sJSP

Analysis of the sJSP

- Different rates ⇒ schedule important
- Synthesis of best and worst schedules



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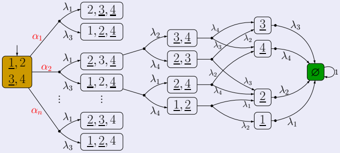
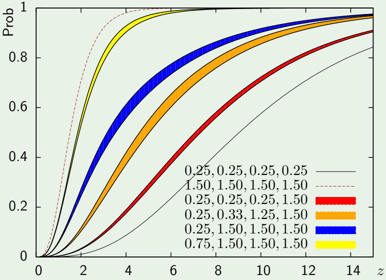


Numerical results: Maximum and minimum probabilities



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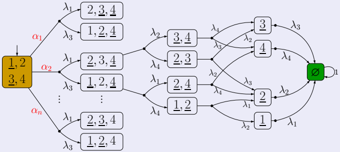

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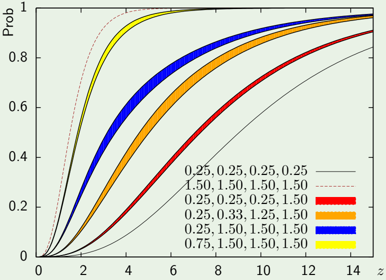

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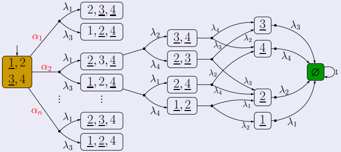

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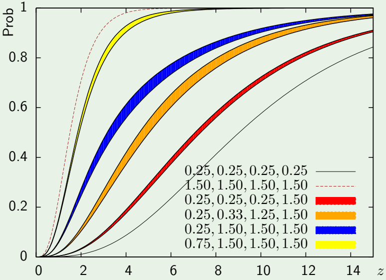


Optimal schedule for $(0.25, 0.33, 1.25, 1.5)$

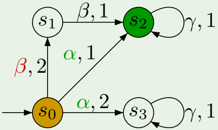
$\underline{1}, \underline{2}, 3, 4$	$1 \mapsto \{2,3\}$ $2 \mapsto \{1,3\}$
$1, \underline{2}, \underline{3}, 4$	$2 \mapsto \{1,3\}$ $3 \mapsto \{1,2\}$
$1, 2, \underline{3}, \underline{4}$	$3 \mapsto \{1,2\}$ $4 \mapsto \{1,2\}$

Outline of the talk

Interactive Markov chains

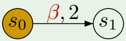
Model Checking Interactive Markov Chains [Zhang, Neuhäußer '10]

Continuous-time MDP



Combines actions and rates.

Only one type of transitions:



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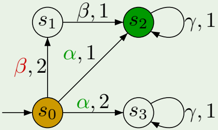

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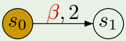
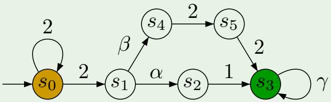
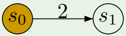


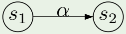
Interactive Markov Chain [Hermanns'02]



Separates actions and rates.

Two types of transitions:

- Markovian 

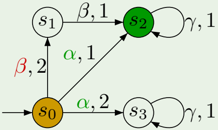
- Interactive 

Maximal progress assumption!

Interactive Markov chains

Model Checking Interactive Markov Chains [Zhang, Neuhäußer '10]

Continuous-time MDP

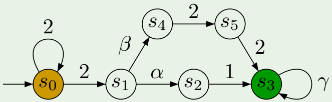


Combines actions and rates.

Only one type of transitions:

$$s_0 \xrightarrow{\beta, 2} s_1$$

Interactive Markov Chain [Hermanns'02]



Separates actions and rates.

Two types of transitions:

- Markovian $s_0 \xrightarrow{2} s_1$
- Interactive $s_1 \xrightarrow{\alpha} s_2$

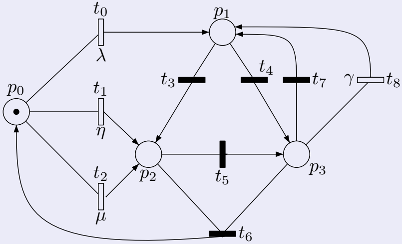
Maximal progress assumption!

Extending the discretization from CTMDPs to IMCs

Model checking the continuous stochastic logic (CSL) on IMCs!

Model checking generalized stochastic Petri nets

Generalized stochastic Petri nets (GSPNs) [Marsan,Conte,Balbo '84]



- Input, output and inhibitor arcs
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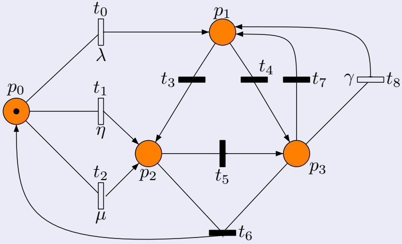

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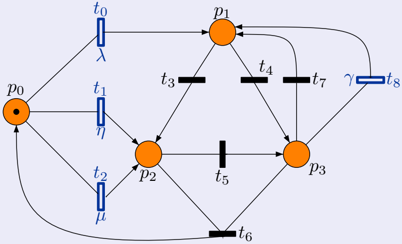

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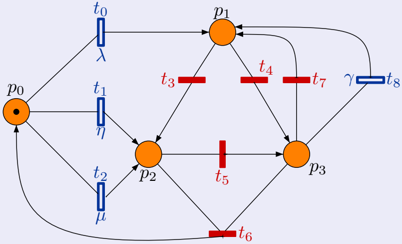

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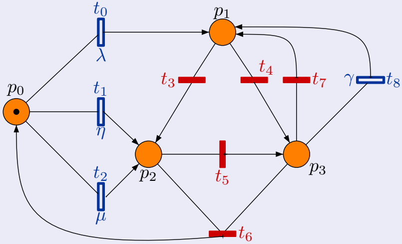

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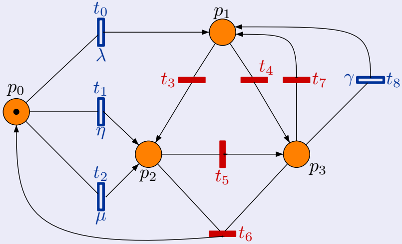

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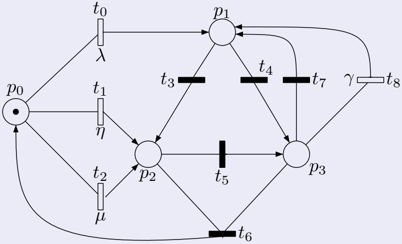

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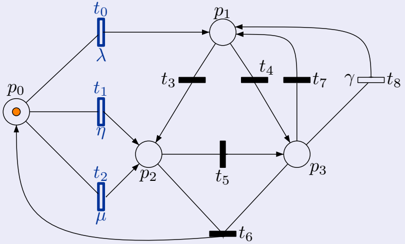

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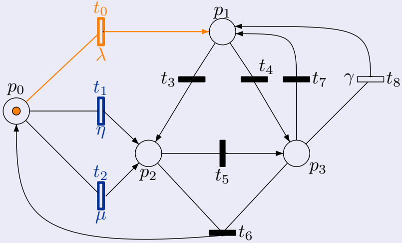

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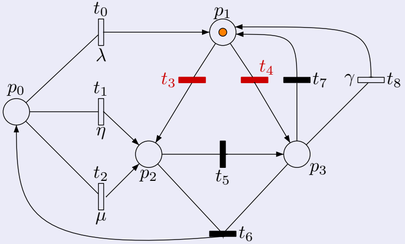

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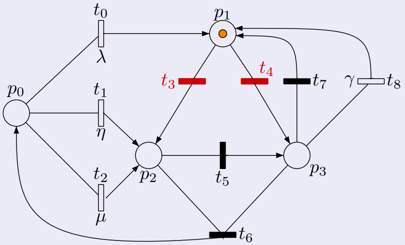

Overcome confusion in GSPNs [Hermanns,Katoen,Neuhäußer,Zhang '10]

Multiple conflicting immediate transitions enabled.
Which one executes first?

- Classical answer: Avoid this case by using weights!
- New approach: Nondeterminism!
  Interpret reachability graph of a GSPN as an IMC!

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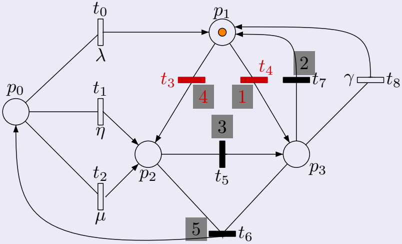

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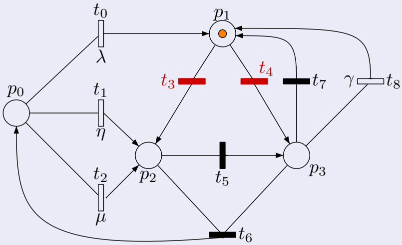

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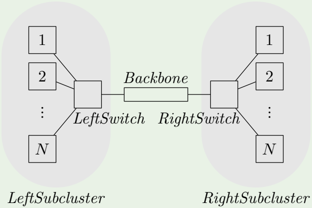
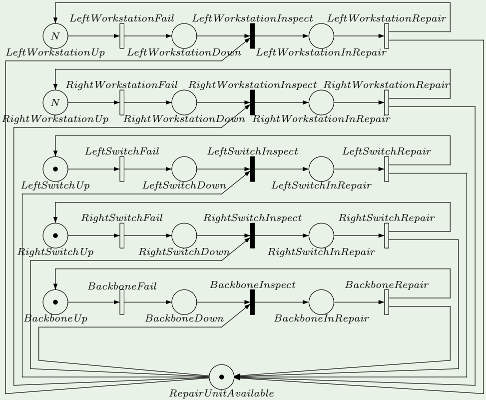
Dependability Analysis of a Workstation Cluster Haverkort'00

A GSPN model for the dependable workstation cluster





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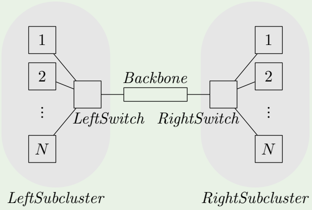
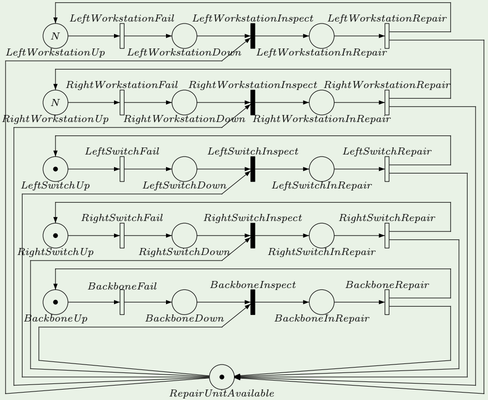




Result of the nondeterministic analysis

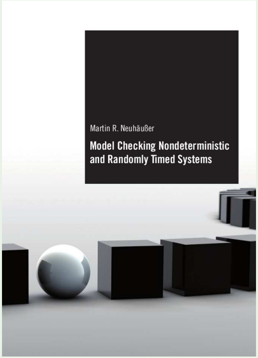
System is 18% less reliable than predicted by earlier analysis!

Outline of the talk

Results of the thesis

What can be found in there?

1. Continuous-time Markov decision processes
   - A new class of time-dependent schedulers
   - Time-bounded reachability analysis
   - Strong bisimulation minimization for CTMDPs.
2. Interactive Markov chains
   - Extension of CTMDP analysis to IMCs
   - CSL model checking algorithm
3. Generalized stochastic Petri nets
   - A simple and concise semantics for GSPNs
   - Model checking CSL formulas on GSPNs
4. Case studies
   - Solving the stochastic job scheduling problem
   - Dependability analysis of a workstation cluster



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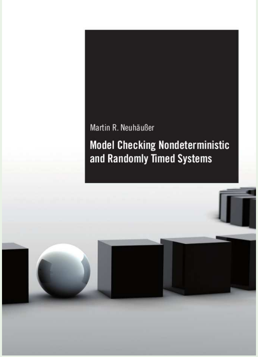


Open problems and future research directions

The future...

1. Continuous-time Markov decision processes
   - Restriction to local uniformity?
   - Uniformization for time-dependent schedulers?
2. Interactive Markov chains
   - Computing long run average measures?
   - Support for reward extensions of CSL?
3. Generalized stochastic Petri nets
   - Allow for partial weight specifications?
   - Extension towards stochastic activity networks?


THANK
YOU

List of Publications

Published papers

1. *Model Checking Interactive Markov Chains.*
   Zhang, Neuhäußer. TACAS 2010
2. *Delayed Nondeterminism in Continuous-Time Markov Decision Processes.*
   Neuhäußer, Stoelinga, Katoen. FoSSaCS 2009
3. *Compositional Abstraction for Stochastic Systems.*
   Katoen, Klink, Neuhäußer. FORMATS 2009
4. *Bisimulation and Logical Preservation for Continuous-Time Markov Decision Processes.*
   Neuhäußer, Katoen. CONCUR 2007
5. *Abstraction and Model Checking of Core Erlang Programs in Maude.*
   Neuhäußer, Noll. WRLA 2007

The pipeline

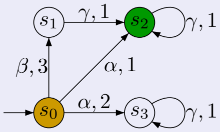
1. *Time-Bounded Reachability in Continuous-Time Markov Decision Processes.*
   Neuhäußer, Zhang. submitted to LICS 2010
2. *GSPN model checking despite confusion.*
   Hermanns, Katoen, Neuhäußer, Zhang. submitted to ICATPN 2010

Are time-dependent schedulers necessary?

Time-abstract vs. time dependent schedulers

- Why not positional schedulers?
- ... or time-abstract schedulers?

Are our schedulers really better?



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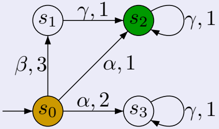
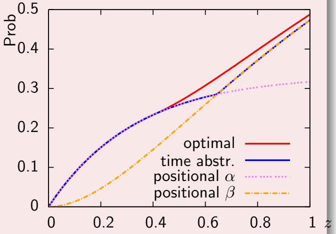


Maximum probability to reach state s_2 in ≤ 1 time unit

Yes, they are!



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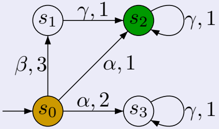
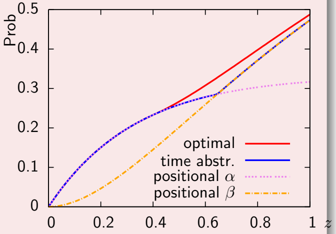


Maximum probability to reach state s_2 in ≤ 1 time unit

Why is this:
Generic scheduler decides upon leaving s_0:

- If long time remains: choose β
- If few time remains: choose α.

Time-abstract schedulers cannot do this!



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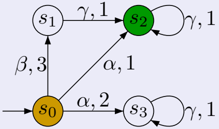

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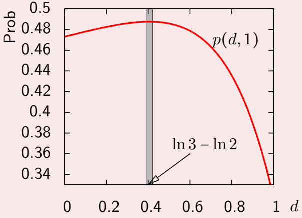

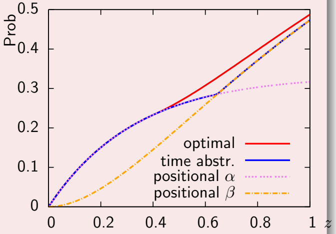

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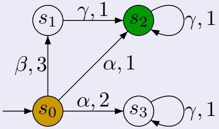

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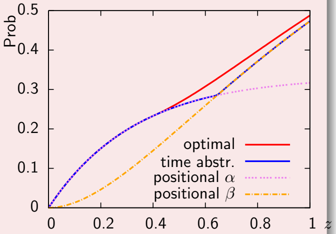
Optimal scheduler for time-bound $z = 1$:

$$D(s_0, t_0) = \begin{cases} \{\alpha \mapsto 1\} & \text{if } (1 - t_0) \leq \ln 3 - \ln 2 \\ \{\beta \mapsto 1\} & \text{otherwise.} \end{cases}$$



A simpler class of optimal schedulers

Total time positional schedulers

A scheduler $D : Paths^\star \times \mathbb{R}_{\geq 0} \to Distr(Act)$ is total time positional iff
$\forall \pi, \pi' \in Paths^\star .\ \forall t, t' \in \mathbb{R}_{\geq 0}$.

$$\Big(last(\pi) = last(\pi') \wedge \Delta(\pi) + t = \Delta(\pi') + t' \Big) \Rightarrow D(\pi, t) = D(\pi', t').$$

$\Delta(\pi)$ is the total time spent on π.

Intuition

Total time positional schedulers only depend on

- the current state $last(\pi)$
- the total amount of time $\Delta(\pi) + t$ that has passed.

Optimality of TTPD schedulers

There exists $D \in TTPD$ such that $Pr^{\omega}_{s,D^*}(\Diamond^{[0,z]} G) = p^{G}_{max}(s, z)$.

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