Static Program Analysis

Lecture 12: Abstract Interpretation I (Theoretical Foundations)

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Outline

Abstract Interpretation I

- **Summary:** a theory of sound approximation of the semantics of programs
- **Basic idea:** execution of program on abstract values (similar to type-level bytecode interpreter)
- **Example:** parity (even/odd) rather than concrete numbers

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- **Basic idea:** execution of program on abstract values
  (similar to type-level bytecode interpreter)
- **Example:** parity (even/odd) rather than concrete numbers
- **Procedure:** run program on finite set of abstract values that cover all concrete inputs using abstract operations that cover all concrete outputs

  $\Longrightarrow$ soundness of approach
- **Preciseness** of information again characterized by partial order

Abstract Interpretation II

- **Advantages:**
  - Abstract interpretation covers conditional branches (`if`/`while`) without further extension
  - Granularity of abstract domain influences precision and complexity of analysis (mutual tradeoff)
  - Numerous variants for different kinds of programs (functional, concurrent, ...)
  - Soundness is guaranteed if abstract operations are determined according to theory

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  - Granularity of abstract domain influences precision and complexity of analysis (mutual tradeoff)
  - Numerous variants for different kinds of programs (functional, concurrent, ...)
  - Soundness is guaranteed if abstract operations are determined according to theory
- **Disadvantages:**
  - Complexity generally higher than with dataflow analysis
  - Automatic derivation of abstract operations can be difficult

Overview (tentative)

1. Theoretical foundations (Galois connections)
2. (Concrete &) abstract semantics of WHILE programs
3. Automatic derivation of abstract semantics
4. Application: verification of 16-bit multiplication
5. Predicate abstraction
6. CEGAR (CounterExample-Guided Abstraction Refinement)

Outline

Galois Connections I

Definition 12.1 (Galois connection)

Let $(L, \sqsubseteq_L)$ and $(M, \sqsubseteq_M)$ be complete lattices. A pair (α, γ) of monotonic functions

$$\alpha : L \to M \qquad \text{and} \qquad \gamma : M \to L$$

is called a Galois connection if

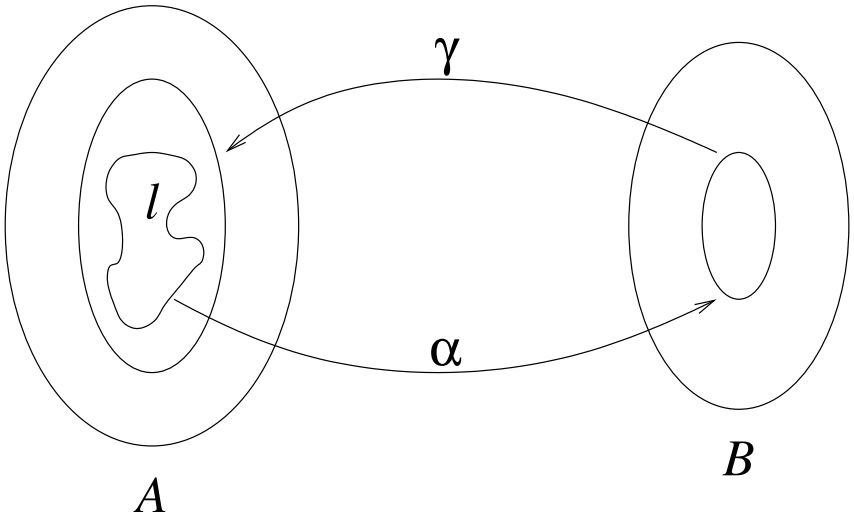
$$\forall l \in L : l \sqsubseteq_L \gamma(\alpha(l)) \qquad \text{and} \qquad \forall m \in M : \alpha(\gamma(m)) \sqsubseteq_M m$$

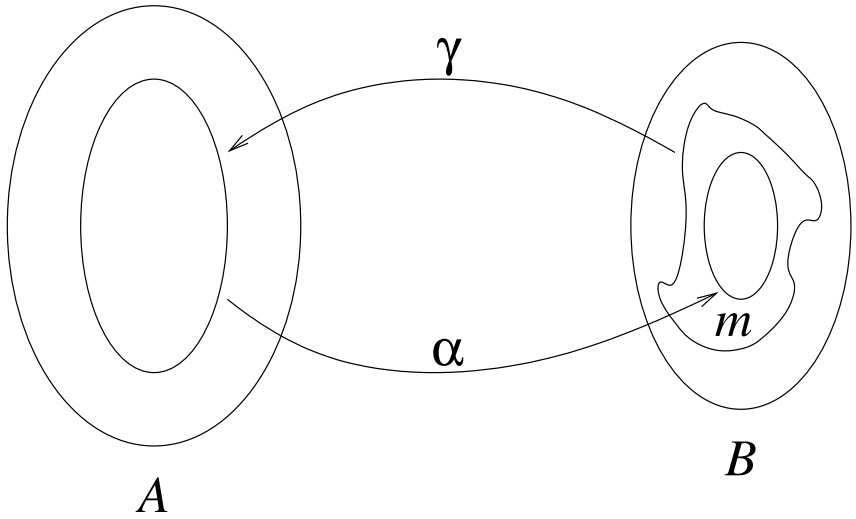
Interpretation:

- $L = \{\text{sets of concrete values}\}$, $M = \{\text{sets of abstract values}\}$
- $\alpha =$ abstraction function, $\gamma =$ concretization function
- $l \sqsubseteq_L \gamma(\alpha(l))$: α yields over-approximation
- $\alpha(\gamma(m)) \sqsubseteq_M m$: no loss of precision by abstraction after concretization
- Usually: $l \neq \gamma(\alpha(l))$, $\alpha(\gamma(m)) = m$

Galois Connections II

For $A = \{\text{concrete values}\}$, $B = \{\text{abstract values}\}$, $L = 2^A$, $M = 2^B$:





$\forall l \in L : l \sqsubseteq_L \gamma(\alpha(l))$

(α yields over-approximation)

$\forall m \in M : \alpha(\gamma(m)) \sqsubseteq_M m$

(no loss of precision by abstraction after concretization)

Example 12.2 (Parity abstraction)

Concrete domain: $L = (2^{\mathbb{Z}}, \subseteq)$ Abstract domain: $M = (2^{\{\text{even},\text{odd}\}}, \subseteq)$

$$\alpha : 2^{\mathbb{Z}} \to 2^{\{\text{even},\text{odd}\}}$$

$$\alpha(Z) := \begin{cases} \emptyset & \text{if } Z = \emptyset \\ \{\text{even}\} & \text{if } Z \subseteq \mathbb{Z}_{\text{even}} \\ \{\text{odd}\} & \text{if } Z \subseteq \mathbb{Z}_{\text{odd}} \\ \{\text{even}, \text{odd}\} & \text{otherwise} \end{cases}$$

$$\gamma : 2^{\{\text{even},\text{odd}\}} \to 2^{\mathbb{Z}}$$

$$\gamma(P) := \bigcup_{p \in P} \mathbb{Z}_p$$

where

$$\mathbb{Z}_{\text{even}} := \{\ldots, -2, 0, 2, \ldots\}$$
$$\mathbb{Z}_{\text{odd}} := \{\ldots, -3, -1, 1, 3, \ldots\}$$

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yields a Galois connection. For example,

- $\gamma(\alpha(\{1,3,7\})) = \gamma(\{\text{odd}\}) = \{\ldots, -3, -1, 1, 3, \ldots\} \supseteq \{1,3,7\}$
- $\alpha(\gamma(\{\text{even}\})) = \alpha(\{\ldots, -2, 0, 2, \ldots\}) = \{\text{even}\}$

Galois Connections IV

Example 12.3 (Sign abstraction)

Concrete domain: $L = (2^{\mathbb{Z}}, \subseteq)$ Abstract domain: $M = (2^{\{+,-,0\}}, \subseteq)$

$$\alpha : 2^{\mathbb{Z}} \to 2^{\{+,-,0\}}$$
$$\alpha(Z) := \{\text{sgn}(z) \mid z \in Z\}$$

$$\gamma : 2^{\{+,-,0\}} \to 2^{\mathbb{Z}}$$
$$\gamma(S) := \bigcup_{s \in S} \mathbb{Z}_s$$

where

$$\text{sgn}(z) := \begin{cases} + & \text{if } z > 0 \\ - & \text{if } z < 0 \\ 0 & \text{otherwise} \end{cases}$$
$$\mathbb{Z}_+ := \{1, 2, 3, \ldots\}$$
$$\mathbb{Z}_- := \{-1, -2, -3, \ldots\}$$
$$\mathbb{Z}_0 := \{0\}$$

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- $\gamma(\alpha(\{0,1,3\})) = \gamma(\{+,0\}) = \{0,1,2,3,\ldots\} \supseteq \{0,1,3\}$
- $\alpha(\gamma(\{+,-\})) = \alpha(\mathbb{Z} \setminus \{0\}) = \{+,-\}$

Example 12.4 (Interval abstraction (cf. Slide 8.5))

Concrete domain: $L = (2^{\mathbb{Z}}, \subseteq)$ Abstract domain: $M = (Int, \subseteq)$

$$\alpha : 2^{\mathbb{Z}} \to Int$$

$$\alpha(Z) := \begin{cases} \emptyset & \text{if } Z = \emptyset \\ [\bigsqcap Z, \bigsqcup Z] & \text{otherwise} \end{cases}$$

$$\gamma : Int \to 2^{\mathbb{Z}}$$

$$\gamma(I) := \begin{cases} \emptyset & \text{if } I = \emptyset \\ \{z \in \mathbb{Z} \mid z_1 \leq z \leq z_2\} & \text{if } I = [z_1, z_2] \end{cases}$$

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Galois Connections V

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yields a Galois connection. For example,

- $\gamma(\alpha(\{1, 3, 5, \ldots\})) = \gamma([1, +\infty]) = \{1, 2, 3, 4, 5, \ldots\} \supseteq \{1, 3, 5, \ldots\}$
- $\alpha(\gamma([-1, 1])) = \alpha(\{-1, 0, 1\}) = [-1, 1]$

Properties of Galois Connections

Lemma 12.5

Let (α, γ) be a Galois connection with $\alpha : L \to M$ and $\gamma : M \to L$, and let $l \in L$, $m \in M$, $L' \subseteq L$, $M' \subseteq M$.

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$$\gamma(m) = \bigsqcup \{ l \in L \mid \alpha(l) \sqsubseteq_M m \}$$

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3. α *is uniquely determined by* γ *as follows:*
$$\alpha(l) = \bigsqcap\{m \in M \mid l \sqsubseteq_L \gamma(m)\}$$
4. α *is completely distributive:*
$$\alpha(\bigsqcup L') = \bigsqcup\{\alpha(l) \mid l \in L'\}$$

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Proof.

on the board □

Outline

Reminder: Syntax of WHILE

The syntax of WHILE Programs is defined by the following context-free grammar (cf. Definition 1.3):

$a ::= z \mid x \mid a_1 \texttt{+} a_2 \mid a_1 \texttt{-} a_2 \mid a_1 \texttt{*} a_2 \in \mathit{AExp}$
$b ::= t \mid a_1 \texttt{=} a_2 \mid a_1 \texttt{>} a_2 \mid \neg b \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \in \mathit{BExp}$
$c ::= \texttt{skip} \mid x \texttt{ := } a \mid c_1 \texttt{;} c_2 \mid \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2 \mid \texttt{while } b \texttt{ do } c \in \mathit{Cmd}$

Program States

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- Depends on the values of the variables in the expression

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Definition 12.6 (Program state)

A (program) state is an element of the set

$$\Sigma := \{\sigma \mid \sigma : \mathit{Var} \to \mathbb{Z}\},$$

called the state space.

Thus $\sigma(x)$ denotes the value of $x \in \mathit{Var}$ in state $\sigma \in \Sigma$.

Evaluation of Expressions

Definition 12.7 (Evaluation function)

Let $\sigma \in \Sigma$ be a state.

1. $val_\sigma : AExp \to \mathbb{Z} : a \to val_\sigma(a)$
   yields the value of a in state σ
2. $val_\sigma : BExp \to \mathbb{B} : b \to val_\sigma(b)$
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Example 12.8

Let $\sigma(\texttt{x}) = 1$ and $\sigma(\texttt{y}) = 2$.

1. $val_\sigma(\texttt{2 * x + y}) = 4$
2. $val_\sigma(\neg\texttt{(x + 1 > y)}) = \text{true}$

Semantics of WHILE I

Definition employs derivation rules of the form

$$\text{Name}\ \frac{\text{Premise(s)}}{\text{Conclusion}}$$

- meaning: if every premise is fulfilled, then conclusion can be drawn
- a rule with no premises is called an axiom

Definition 12.9 (Execution relation for statements)

If $c \in \mathit{Cmd}$ and $\sigma \in \Sigma$, then $\langle c, \sigma \rangle$ is called a configuration. The execution relation (on configurations and states) is defined by the following rules:

$$(\text{skip})\ \frac{}{\langle \texttt{skip}, \sigma \rangle \to \sigma}$$

$$(\text{asgn})\ \frac{}{\langle x\ \texttt{:=}\ a, \sigma \rangle \to \sigma[x \mapsto \mathit{val}_\sigma(a)]}$$

$$(\text{seq1})\ \frac{\langle c_1, \sigma \rangle \to \langle c_1', \sigma' \rangle}{\langle c_1\,;c_2, \sigma \rangle \to \langle c_1'\,;c_2, \sigma' \rangle}$$

$$(\text{seq2})\ \frac{\langle c_1, \sigma \rangle \to \sigma'}{\langle c_1\,;c_2, \sigma \rangle \to \langle c_2, \sigma' \rangle}$$

Semantics of WHILE III

Definition 12.9 (Execution relation for statements; continued)

$$(\text{if1})\ \frac{val_\sigma(b) = \text{true}}{\langle \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2, \sigma\rangle \to \langle c_1, \sigma\rangle}$$

$$(\text{if2})\ \frac{val_\sigma(b) = \text{false}}{\langle \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2, \sigma\rangle \to \langle c_2, \sigma\rangle}$$

$$(\text{wh1})\ \frac{val_\sigma(b) = \text{true}}{\langle \texttt{while } b \texttt{ do } c, \sigma\rangle \to \langle c \texttt{;while } b \texttt{ do } c, \sigma\rangle}$$

$$(\text{wh2})\ \frac{val_\sigma(b) = \text{false}}{\langle \texttt{while } b \texttt{ do } c, \sigma\rangle \to \sigma}$$