# Static Program Analysis

## Lecture 17: Abstract Interpretation VI (Predicate Abstraction)

Thomas Noll

Lehrstuhl für Informatik 2
(Software Modeling and Verification)

RWTH Aachen University

noll@cs.rwth-aachen.de

http://www-i2.informatik.rwth-aachen.de/i2/spa11/

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# Informatik Sommerfest

**Im Informatikzentrum**
**Ahornstraße 55**

**1. Juli 2011**


16:00 **Beginn**

16:30 **Festakt** (Aula 2)
**Verleihung der Zeugnisse**

danach **Geselliges Feiern**
(Parkplatz, Foyer E2)

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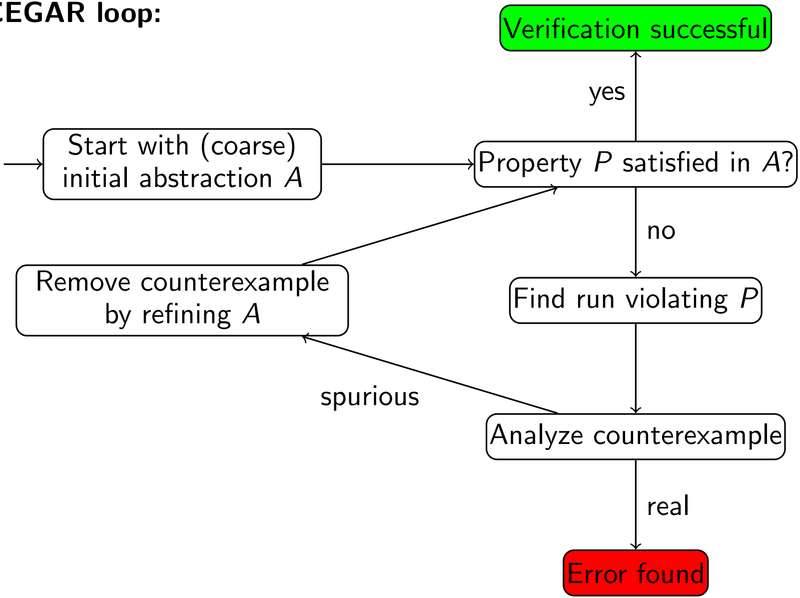
# Outline

# Abstraction Refinement

- **Problem:** desired program property cannot be shown using current abstraction method
- **Reasons:**
  1. program really violates property or
  2. current abstraction is too coarse
- **Solutions:**
  1. fix the problem
  2. refine abstraction
- **Abstraction refinement:** most successful (automatic) method based on
  - predicate abstraction and
  - counterexamples

**CEGAR loop:**


Verification successful
yes
Start with (coarse) initial abstraction $A$
Property $P$ satisfied in $A$?
no
Remove counterexample by refining $A$
Find run violating $P$
spurious
Analyze counterexample
real
Error found

# Abstraction Refinement for Predicates

1. Extract predicates (i.e., logical formulae) from counterexample
2. Use Galois connection that classifies program states according to validity of predicates (predicate abstraction)
3. Compute new abstract semantics and search for new counterexamples
4. Iterate until property satisfied or real counterexample found (with increasing set of predicates)

# Outline

## Definition 17.1 (Predicate abstraction)

Let *Var* be a set of variables.

- A predicate is a Boolean expression $p \in BExp$ over *Var*.
- A state $\sigma \in \Sigma$ satisfies $p \in BExp$ ($\sigma \models p$) if $val_\sigma(p) = \text{true}$.
- $p$ implies $q$ ($p \models q$) if $\sigma \models q$ whenever $\sigma \models p$ (or: $p$ is stronger than $q$, $q$ is weaker than $p$).
- $p$ and $q$ are equivalent ($p \equiv q$) if $p \models q$ and $q \models p$.
- Let $P = \{p_1, \ldots, p_n\} \subseteq BExp$ be a finite set of predicates, and let $\neg P := \{\neg p_1, \ldots, \neg p_n\}$. An element of $P \cup \neg P$ is called a literal. The predicate abstraction lattice is defined by:

$$Abs(p_1, \ldots, p_n) := \left(\left\{\bigwedge Q \mid Q \subseteq P \cup \neg P\right\}, \models\right).$$

**Abbreviations:** $\text{true} := \bigwedge \emptyset$, $\text{false} := \ldots \wedge p_i \wedge \ldots \wedge \neg p_i \wedge \ldots$

# Predicate Abstraction II

## Lemma 17.2

*$Abs(p_1, \ldots, p_n)$ is a* *complete lattice* *with*

- $\bot = \text{false}$, $\top = \text{true}$
- $q_1 \sqcap q_2 = q_1 \wedge q_2$
- $q_1 \sqcup q_2 = \overline{q_1 \vee q_2}$ *where* $\overline{b} := \bigwedge\{q \in Abs(p_1, \ldots, p_n) \mid b \models q\}$
  *(i.e., strongest formula in $Abs(p_1, \ldots, p_n)$ that is implied by $q_1 \vee q_2$)*

## Example 17.3

Let $P := \{p_1, p_2, p_3\}$.

1. For $q_1 := p_1 \wedge \neg p_2$ and $q_2 := \neg p_2 \wedge p_3$, we obtain
$$q_1 \sqcap q_2 = q_1 \wedge q_2 \equiv p_1 \wedge \neg p_2 \wedge p_3$$
$$q_1 \sqcup q_2 = \overline{q_1 \vee q_2} \equiv \overline{\neg p_2 \wedge (p_1 \vee p_3)} \equiv \neg p_2$$
2. For $q_1 := p_1 \wedge p_2$ and $q_2 := p_1 \wedge \neg p_2$, we obtain
$$q_1 \sqcap q_2 = q_1 \wedge q_2 \equiv \text{false}$$
$$q_1 \sqcup q_2 = \overline{q_1 \vee q_2} \equiv \overline{p_1 \wedge (p_2 \vee \neg p_2)} \equiv p_1$$

# Predicate Abstraction III

## Definition 17.4 (Galois connection for predicate abstraction)

The Galois connection for predicate abstraction is determined by

$$\alpha : 2^\Sigma \to Abs(p_1, \ldots, p_n) \quad \text{and} \quad \gamma : Abs(p_1, \ldots, p_n) \to 2^\Sigma$$

with

$$\alpha(S) := \bigsqcup\{q_\sigma \mid \sigma \in S\} \quad \text{and} \quad \gamma(q) := \{\sigma \in \Sigma \mid \sigma \models q\}$$

where $q_\sigma := \bigwedge(\{p_i \mid 1 \leq i \leq n, \sigma \models p_i\} \cup \{\neg p_i \mid 1 \leq i \leq n, \sigma \not\models p_i\})$.

## Example 17.5

- Let $Var := \{\mathtt{x}, \mathtt{y}\}$
- Let $P := \{p_1, p_2, p_3\}$ where $p_1 := (\mathtt{x<=y})$, $p_2 := (\mathtt{x=y})$, $p_3 := (\mathtt{x>y})$
- If $S = \{\sigma_1, \sigma_2\} \subseteq \Sigma$ with $\sigma_1 = [\mathtt{x} \mapsto 1, \mathtt{y} \mapsto 2]$, $\sigma_2 = [\mathtt{x} \mapsto 2, \mathtt{y} \mapsto 2]$, then
$$\begin{aligned} \alpha(S) &= q_{\sigma_1} \sqcup q_{\sigma_2} \\ &= \underline{(p_1 \wedge \neg p_2 \wedge \neg p_3) \sqcup (p_1 \wedge p_2 \wedge \neg p_3)} \\ &\equiv (p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge \neg p_3) \\ &\equiv p_1 \wedge \neg p_3 \end{aligned}$$
- If $q = p_1 \wedge \neg p_2 \in Abs(p_1, \ldots, p_n)$, then $\gamma(q) = \{\sigma \in \Sigma \mid \sigma(\mathtt{x}) < \sigma(\mathtt{y})\}$

# Outline

## Definition 17.6 (Execution relation for predicate abstraction)

If $c \in Cmd$ and $q \in Abs(p_1, \ldots, p_n)$, then $\langle c, q \rangle$ is called an abstract configuration. The execution relation for predicate abstraction is defined by the following rules:

$$\text{(skip)}\ \frac{}{\langle \texttt{skip}, q \rangle \Rightarrow q} \qquad \text{(asgn)}\ \frac{}{\langle x \texttt{ := } a, q \rangle \Rightarrow \bigsqcup\{q_{\sigma[x \mapsto val_\sigma(a)]} \mid \sigma \models q\}}$$

$$\text{(seq1)}\ \frac{\langle c_1, q \rangle \Rightarrow \langle c_1', q' \rangle}{\langle c_1 \texttt{;} c_2, q \rangle \Rightarrow \langle c_1' \texttt{;} c_2, q' \rangle} \qquad \text{(seq2)}\ \frac{\langle c_1, q \rangle \Rightarrow q'}{\langle c_1 \texttt{;} c_2, q \rangle \Rightarrow \langle c_2, q' \rangle}$$

$$\text{(if1)}\ \frac{}{\langle \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2, q \rangle \Rightarrow \langle c_1, \overline{q \wedge b} \rangle}$$

$$\text{(if2)}\ \frac{}{\langle \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2, q \rangle \Rightarrow \langle c_2, \overline{q \wedge \neg b} \rangle}$$

$$\text{(wh1)}\ \frac{}{\langle \texttt{while}\ b\ \texttt{do}\ c, q \rangle \Rightarrow \langle c \texttt{;} \texttt{while}\ b\ \texttt{do}\ c, \overline{q \wedge b} \rangle}$$

$$\text{(wh2)}\ \frac{}{\langle \texttt{while}\ b\ \texttt{do}\ c, q \rangle \Rightarrow \overline{q \wedge \neg b}}$$

**Remarks:**

- In Rule (asgn), $\bigsqcup\{q_{\sigma[x\mapsto val_\sigma(a)]} \mid \sigma \models q\}$ denotes the strongest postcondition of $q$ w.r.t. statement $x$ := $a$. It covers all states that are obtained from a state satisfying $q$ by applying the assignment $x$ := $a$:

$$\begin{array}{llcl} \text{Abstract:} & \langle x := a, q\rangle & \Rightarrow & \bigsqcup\{q_{\sigma[x\mapsto val_\sigma(a)]} \mid \sigma \models q\} \\ & \quad\downarrow\gamma & & \quad\uparrow\alpha \\ \text{Concrete:} & \langle x := a, \{\sigma \in \Sigma \mid \sigma \models q\}\rangle & \to & \{\sigma[x \mapsto val_\sigma(a)] \mid \sigma \models q\} \end{array}$$

- In Rules (if1, (if2), (wh1), (wh2): if $b = p_i$ for some $i \in \{1, \ldots, n\}$, then $q \wedge [\neg]b \in Abs(p_1, \ldots, p_n)$, and thus $\overline{q \wedge [\neg]b} = q \wedge [\neg]b$
- An abstract configuration of the form $\langle c, \text{false}\rangle$ represents an unreachable configuration (as there is no $\sigma \in \Sigma$ such that $\sigma \models \text{false}$) and can therefore be omitted
- If $P = \emptyset$ (and thus $Abs(P) = \{\text{true}, \text{false}\}$) and if no $b \in BExp_c$ is a contradiction, then the abstract transition system corresponds to the control flow graph of $c$

# Abstract Semantics for Predicate Abstraction III

## Example 17.7

```
if [x > y]^1 then
  while [¬(y = 0)]^2 do
    [x := x - 1;]^3;
    [y := y - 1;]^4;
  if [x > y]^5 then
    [skip]^6;
  else
    [skip]^7;
else
  [skip]^8;
```

- **Claim:** label 7 not reachable (as `x > y` is a loop invariant)
- **Proof:** by predicate abstraction with $p_1 := ($`x > y`$)$ and $p_2 := ($`x >= y`$)$
- **Abstract transition system:** on the board
- **Remark:** $p_1 := ($`x > y`$)$ alone not sufficient (as not necessarily valid after label 3)

# Abstract Semantics for Predicate Abstraction IV

**Problem:** $q'$ generally not computable in

$$\text{(asgn)}\ \frac{}{\langle x := a, q\rangle \Rightarrow \underbrace{\bigsqcup\{q_{\sigma[x\mapsto val_\sigma(a)]} \mid \sigma \models q\}}_{q'}}$$

(due to undecidability of implication in certain logics)

**Solutions:**

- Over-approximation: fall back to non-strongest postconditions
  - in practice, (automatic) theorem proving
  - for every $i \in \{1, \ldots, n\}$, try to prove $q' \models p_i$ and $q' \models \neg p_i$
  - approximate $q'$ by conjunction of all provable literals
- Restriction of programs:
  - $\models$ decidable for certain logics
  - example: Presburger arithmetic (first-order theory of $\mathbb{N}$ with $+$)
  - thus $q'$ computable for WHILE programs without multiplication
- Restriction to finite domains:
  - for example, binary numbers of fixed size
  - thus everything (domain, Galois connection, ...) exactly computable
  - problem: exponential blowup $\Longrightarrow$ solution: Binary Decision Diagrams