Static Program Analysis

Lecture 2: Dataflow Analysis I (Introduction & Available Expressions Analysis)

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Outline

Dataflow Analysis: the Approach

- Traditional form of program analysis

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- Idea: describe how analysis information flows through program

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- Idea: describe how analysis information flows through program
- Distinctions:

direction of flow:
: forward vs. backward analyses

procedures:
: interprocedural vs. intraprocedural analyses

quantification over paths:
: may (union) vs. must (intersection) analyses

dependence on statement order:
: flow-sensitive vs. flow-insensitive analyses

Labeled Programs

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  - `skip` statements
  - assignments
  - tests in conditionals (`if`) and loops (`while`)

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  - assignments
  - tests in conditionals (`if`) and loops (`while`)
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Definition 2.1 (Labeled WHILE programs)

The syntax of labeled WHILE programs is defined by the following context-free grammar:

$$
\begin{aligned}
a &::= z \mid x \mid a_1 \texttt{+} a_2 \mid a_1 \texttt{-} a_2 \mid a_1 \texttt{*} a_2 \in AExp \\
b &::= t \mid a_1 \texttt{=} a_2 \mid a_1 \texttt{>} a_2 \mid \neg b \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \in BExp \\
c &::= [\texttt{skip}]^l \mid [x \texttt{ := } a]^l \mid c_1 \texttt{;} c_2 \mid \\
&\qquad \texttt{if } [b]^l \texttt{ then } c_1 \texttt{ else } c_2 \mid \texttt{while } [b]^l \texttt{ do } c \in Cmd
\end{aligned}
$$

- All labels in $c \in Cmd$ assumed distinct, denoted by L_c
- Labeled fragments of c called blocks, denoted by Blk_c

A WHILE Program

Example 2.2

```
x := 6;
y := 7;
z := 0;
while x > 0 do
  x := x - 1;
  v := y;
  while v > 0 do
    v := v - 1;
    z := z + 1;
```

A WHILE Program with Labels

Example 2.2

```
[x := 6]^1;
[y := 7]^2;
[z := 0]^3;
while [x > 0]^4 do
  [x := x - 1]^5;
  [v := y]^6;
  while [v > 0]^7 do
    [v := v - 1]^8;
    [z := z + 1]^9
```

Representing Control Flow I

Every (labeled) statement has a single entry (given by the initial label) and generally multiple exits (given by the final labels):

Definition 2.3 (Initial and final labels)

The mapping init : $Cmd \to L$ returns the initial label of a statement:

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\begin{aligned}
\text{init}([\texttt{skip}]^l) &:= l \\
\text{init}([x \texttt{ := } a]^l) &:= l \\
\text{init}(c_1 \texttt{;} c_2) &:= \text{init}(c_1) \\
\text{init}(\texttt{if } [b]^l \texttt{ then } c_1 \texttt{ else } c_2) &:= l \\
\text{init}(\texttt{while } [b]^l \texttt{ do } c) &:= l
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\end{aligned}$$

The mapping final : $Cmd \to 2^L$ returns the set of final labels of a statement:

$$\begin{aligned}
\text{final}([\texttt{skip}]^l) &:= \{l\}\\
\text{final}([x \texttt{ := } a]^l) &:= \{l\}\\
\text{final}(c_1 \texttt{;} c_2) &:= \text{final}(c_2)\\
\text{final}(\texttt{if } [b]^l \texttt{ then } c_1 \texttt{ else } c_2) &:= \text{final}(c_1) \cup \text{final}(c_2)\\
\text{final}(\texttt{while } [b]^l \texttt{ do } c) &:= \{l\}
\end{aligned}$$

Representing Control Flow II

Definition 2.4 (Flow relation)

Given a statement $c \in Cmd$, the (control) flow relation $\mathrm{flow}(c) \subseteq L \times L$ is defined by

$$
\begin{aligned}
\mathrm{flow}([\texttt{skip}]^l) &:= \emptyset \\
\mathrm{flow}([x \texttt{ := } a]^l) &:= \emptyset \\
\mathrm{flow}(c_1 \texttt{;} c_2) &:= \mathrm{flow}(c_1) \cup \mathrm{flow}(c_2) \cup \\
&\quad \{(l, \mathrm{init}(c_2)) \mid l \in \mathrm{final}(c_1)\} \\
\mathrm{flow}(\texttt{if } [b]^l \texttt{ then } c_1 \texttt{ else } c_2) &:= \mathrm{flow}(c_1) \cup \mathrm{flow}(c_2) \cup \\
&\quad \{(l, \mathrm{init}(c_1)), (l, \mathrm{init}(c_2))\} \\
\mathrm{flow}(\texttt{while } [b]^l \texttt{ do } c) &:= \mathrm{flow}(c) \cup \{(l, \mathrm{init}(c))\} \cup \\
&\quad \{(l', l) \mid l' \in \mathrm{final}(c)\}
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Example 2.5

$c = $ [z := 1]^1;
    while [x > 0]^2 do
        [z := z*y]^3;
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Representing Control Flow III

Example 2.5

$c = [$`z := 1`$]^1$`;`
```
    while [x > 0]^2 do
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```

$$\begin{aligned} \text{init}(c) &= 1 \\ \text{final}(c) &= \{2\} \\ \text{flow}(c) &= \{(1,2),(2,3),(3,4),(4,2)\} \end{aligned}$$

Representing Control Flow III

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$c = $ [z := 1]^1;
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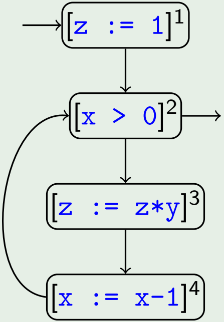
Visualization by (control) flow graph:



Representing Control Flow IV

- To simplify the presentation we will often assume that the program $c \in Cmd$ under consideration has an isolated entry, meaning that
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(which is the case when c does not start with a `while` loop)

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- Similarly: $c \in \mathit{Cmd}$ has isolated exits if
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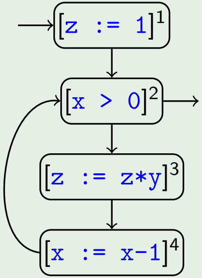
Example 2.6



has an isolated entry but not isolated exits

Outline

1. Preliminaries on Dataflow Analysis
2. An Example: Available Expressions Analysis

Goal of Available Expressions Analysis

Available Expressions Analysis

The goal of Available Expressions Analysis is to determine, for each program point, which (complex) expressions *must* have been computed, and not later modified, on all paths to the program point.

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Example 2.7 (Available Expressions Analysis)

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[x := a+b]^1;
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- a+b available at label 3
- a+b not available at label 5
- possible optimization:
  `while [y > x]^3 do`

Formalizing Available Expressions Analysis I

- Given $a \in AExp$, $b \in BExp$, $c \in Cmd$
  - $Var_a/Var_b/Var_c$ denotes the set of all variables occurring in $a/b/c$
  - $AExp_b/AExp_c$ denote the sets of all complex arithmetic expressions occurring in b/c

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Formalizing Available Expressions Analysis II

Example 2.8 ($\text{kill}_{\text{AE}}/\text{gen}_{\text{AE}}$ functions)

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where $\varphi_{l'} : 2^{AExp_c} \to 2^{AExp_c}$ denotes the transfer function of block $B^{l'}$, given by

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  must: $\bigcap$ in equation for AE_l

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- Later: solution not necessarily unique

  $\Longrightarrow$ choose greatest one

The Equation System II

Reminder:

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Equations:

$$\begin{aligned}
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\mathsf{AE}_2 &= \varphi_1(\mathsf{AE}_1) = \mathsf{AE}_1 \cup \{\texttt{a+b}\} \\
\mathsf{AE}_3 &= \varphi_2(\mathsf{AE}_2) \cap \varphi_5(\mathsf{AE}_5) \\
&= (\mathsf{AE}_2 \cup \{\texttt{a*b}\}) \cap (\mathsf{AE}_5 \cup \{\texttt{a+b}\}) \\
\mathsf{AE}_4 &= \varphi_3(\mathsf{AE}_3) = \mathsf{AE}_3 \cup \{\texttt{a+b}\} \\
\mathsf{AE}_5 &= \varphi_4(\mathsf{AE}_4) = \mathsf{AE}_4 \setminus \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\}
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$$\begin{aligned} \mathrm{AE}_1 &= \emptyset \\ \mathrm{AE}_2 &= \varphi_1(\mathrm{AE}_1) = \mathrm{AE}_1 \cup \{\texttt{a+b}\} \\ \mathrm{AE}_3 &= \varphi_2(\mathrm{AE}_2) \cap \varphi_5(\mathrm{AE}_5) \\ &= (\mathrm{AE}_2 \cup \{\texttt{a*b}\}) \cap (\mathrm{AE}_5 \cup \{\texttt{a+b}\}) \\ \mathrm{AE}_4 &= \varphi_3(\mathrm{AE}_3) = \mathrm{AE}_3 \cup \{\texttt{a+b}\} \\ \mathrm{AE}_5 &= \varphi_4(\mathrm{AE}_4) = \mathrm{AE}_4 \setminus \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\} \end{aligned}$$

Solution:

$$\begin{aligned} \mathrm{AE}_1 &= \emptyset \\ \mathrm{AE}_2 &= \{\texttt{a+b}\} \\ \mathrm{AE}_3 &= \{\texttt{a+b}\} \\ \mathrm{AE}_4 &= \{\texttt{a+b}\} \\ \mathrm{AE}_5 &= \emptyset \end{aligned}$$