

Static Program Analysis

Lecture 9: Dataflow Analysis VIII

(Narrowing & DFA with Conditional Branches)

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Summer Semester 2011

- 1 Repetition: Widening
- 2 Narrowing
- 3 Taking Conditional Branches into Account
- 4 Constant Propagation Analysis with Assertions
- 5 Interval Analysis with Assertions

Definition (Widening operator)

Let $(D, \sqsubseteq)$ be a complete lattice. A mapping $\nabla : D \times D \rightarrow D$ is called **widening operator** if

- for every $d_1, d_2 \in D$,

$$d_1 \sqcup d_2 \sqsubseteq d_1 \nabla d_2$$

and

- for all ascending chains $d_0 \sqsubseteq d_1 \sqsubseteq \dots$, the ascending chain $d_0^\nabla \sqsubseteq d_1^\nabla \sqsubseteq \dots$ eventually stabilizes where

$$d_0^\nabla := d_0 \text{ and } d_{i+1}^\nabla := d_i^\nabla \nabla d_{i+1} \text{ for } i \in \mathbb{N}$$

Remarks:

- $(d_i^\nabla)_{i \in \mathbb{N}}$ is clearly an ascending chain as

$$d_{i+1}^\nabla = d_i^\nabla \nabla d_{i+1} \sqsupseteq d_i^\nabla \sqcup d_{i+1} \sqsupseteq d_i^\nabla$$

- In contrast to $\sqcup$, ∇ does not have to be commutative, associative, monotonic, nor absorptive ($d \nabla d = d$)
- The requirement $d_1 \sqcup d_2 \sqsubseteq d_1 \nabla d_2$ guarantees **soundness** of widening

Applying Widening to Interval Analysis

- A **widening operator**: $\nabla : Int \times Int \rightarrow Int$ with

$$\emptyset \nabla I := I \nabla \emptyset := I$$

$$[x_1, x_2] \nabla [y_1, y_2] := [z_1, z_2] \quad \text{where}$$

$$z_1 := \begin{cases} x_1 & \text{if } x_1 \leq y_1 \\ -\infty & \text{otherwise} \end{cases}$$

$$z_2 := \begin{cases} x_2 & \text{if } y_2 \leq x_2 \\ +\infty & \text{otherwise} \end{cases}$$

- Widening turns infinite ascending chain

$$I_0 = \emptyset \subseteq I_1 = [1, 1] \subseteq I_2 = [1, 2] \subseteq I_3 = [1, 3] \subseteq \dots$$

into a finite one:

$$I_0^\nabla = I_0 = \emptyset$$

$$I_1^\nabla = I_0^\nabla \nabla I_1 = \emptyset \nabla [1, 1] = [1, 1]$$

$$I_2^\nabla = I_1^\nabla \nabla I_2 = [1, 1] \nabla [1, 2] = [1, +\infty]$$

$$I_3^\nabla = I_2^\nabla \nabla I_3 = [1, +\infty] \nabla [1, 3] = [1, +\infty]$$

- In fact, the maximal chain length arising with this operator is 4:

$$\emptyset \subseteq [3, 7] \subseteq [3, +\infty] \subseteq [-\infty, +\infty]$$

Worklist Algorithm with Widening

Goal: extend Algorithm 5.2 by widening to ensure termination

Algorithm (Worklist algorithm with widening)

Input: *dataflow system* $S = (L, E, F, (D, \sqsubseteq), \iota, \varphi)$

Variables: $W \in (L \times L)^*, \{AI_I \in D \mid I \in L\}$

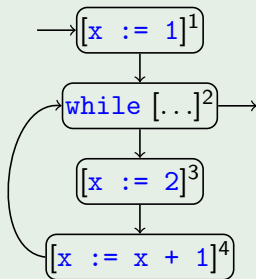
Procedure: $W := \varepsilon$; **for** $(I, I') \in F$ **do** $W := W \cdot (I, I')$; % Initialize W
for $I \in L$ **do** % Initialize AI
 if $I \in E$ **then** $AI_I := \iota$ **else** $AI_I := \perp_D$;
 while $W \neq \varepsilon$ **do**
 $(I, I') := \text{head}(W)$; $W := \text{tail}(W)$;
 if $\varphi_I(AI_I) \not\sqsubseteq AI_{I'}$ **then** % Fixpoint not yet reached
 $AI_{I'} := AI_{I'} \nabla \varphi_I(AI_I)$;
 for $(I', I'') \in F$ **do**
 if (I', I'') not in W **then** $W := (I', I'') \cdot W$;

Output: $\{AI_I \mid I \in L\}$, denoted by $\text{fix}^\nabla(\Phi_S)$

Remark: due to widening, only $\text{fix}(\Phi_S) \sqsubseteq \text{fix}^\nabla(\Phi_S)$ is guaranteed
(cf. Thm. 5.4)

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Example 9.1



Transfer functions (for $\delta(\mathbf{x}) = I$):

$$\varphi_1(I) = [1, 1]$$

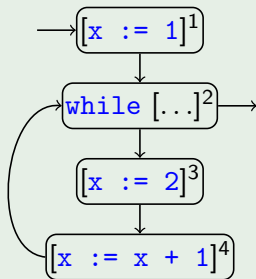
$$\varphi_2(I) = I$$

$$\varphi_3(I) = [2, 2]$$

$$\varphi_4(\emptyset) = \emptyset$$

$$\varphi_4([x_1, x_2]) = [x_1 + 1, x_2 + 1]$$

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Application of worklist algorithm

- ① without widening (omitted):
terminates with expected result for AI_2 ($[1, 3]$)
- ② with widening (on the board):
terminates with unexpected result for AI_2 ($[1, +\infty]$)

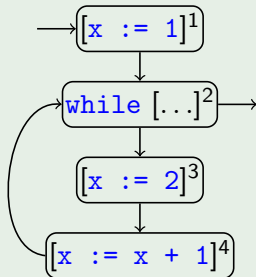
- **Observation:** widening can lead to unnecessarily imprecise results

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- **Solution:** improvement by iterating again from the result obtained by widening (i.e., from $\text{fix}^\nabla(\Phi_S)$)
 $\Rightarrow$ compute $\Phi_S^k(\text{fix}^\nabla(\Phi_S))$ for $k = 1, 2, \dots$

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 $\implies$ compute $\Phi_S^k(\text{fix}^\nabla(\Phi_S))$ for $k = 1, 2, \dots$
- **Soundness:** $\text{fix}(\Phi_S) \sqsubseteq \text{fix}^\nabla(\Phi_S)$
 $\implies \text{fix}(\Phi_S) = \Phi_S^k(\text{fix}(\Phi_S)) \sqsubseteq \Phi_S^k(\text{fix}^\nabla(\Phi_S))$
(since Φ_S and thus Φ_S^k monotonic)

Narrowing Example

Example 9.2 (cf. Example 9.1)



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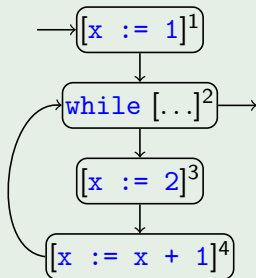
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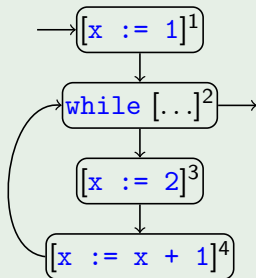
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Narrowing:

	AI_1	AI_2	AI_3	AI_4
$\text{fix}^\nabla(\Phi_S)$	$[-\infty, +\infty]$	$[1, +\infty]$	$[1, +\infty]$	$[2, 2]$

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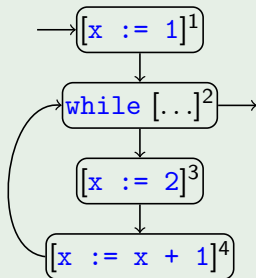
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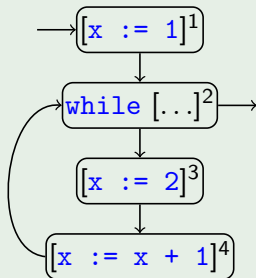
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$\Phi_S^2(\text{fix}^\nabla(\Phi_S))$	$[-\infty, +\infty]$	$[1, 3]$	$[1, 3]$	$[2, 2]$
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(due to infinite descending chains)
- **But:** possible to stop after every step without losing soundness
- **In practice:** termination often ensured by using narrowing operators
($\approx$ counterpart of widening operator; definition omitted)

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Taking Conditional Branches into Account I

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Example 9.3

```
y := 0;  
z := 0;  
while [x > 0]' do  
  if y < 17 then  
    y := y + 1  
  z := z + x;  
  x := x - 1;
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- Interval analysis (with widening) yields for l :

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```
x ∈  $[-\infty, +\infty]$   
y ∈  $[0, +\infty]$   
z ∈  $[-\infty, +\infty]$ 
```

Taking Conditional Branches into Account I

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```

$$\begin{aligned}x &\in [-\infty, +\infty] \\y &\in [0, +\infty] \\z &\in [-\infty, +\infty]\end{aligned}$$

- Too pessimistic! In fact,

$$\begin{aligned}x &\in [-\infty, +\infty] \\y &\in [0, 17] \\z &\in [0, +\infty]\end{aligned}$$

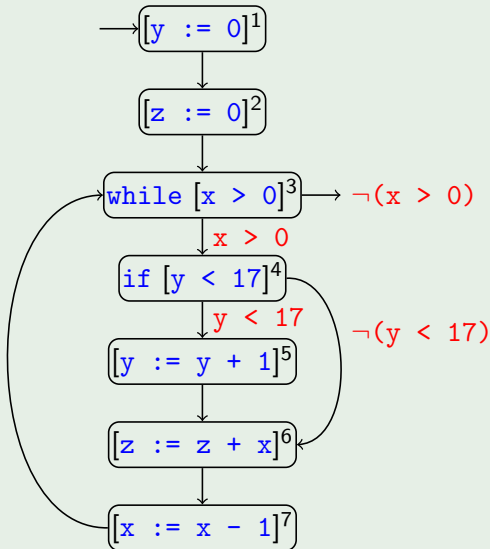
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- **Solution:** introduce **transfer functions for branches**
- **First approach:** attach (negated) conditions as **labels to control flow edges**
 - advantage: no language modification required
 - disadvantage: entails extension of DFA framework
 - will not be considered here
- **Second approach:** encode conditions as **assertions** (statements)
 - advantage: DFA framework can be reused
 - disadvantage: entails extension of WHILE language
 - the way we will follow

First Approach: Conditions as Edge Labels

Example 9.4 (cf. Example 9.3)



Example 9.5 (cf. Example 9.3)

```
y := 0;  
z := 0;  
while x > 0 do  
  assert x > 0;  
  if y < 17 then  
    assert y < 17;  
    y := y + 1;  
  z := z + x;  
  x := x - 1;  
assert  $\neg(x > 0)$ ;
```

Extending the Syntax of WHILE Programs

Definition 9.6 (Labeled WHILE programs with assertions)

The **syntax of labeled WHILE programs with assertions** is defined by the following context-free grammar:

$$\begin{aligned} a &::= z \mid x \mid a_1 + a_2 \mid a_1 - a_2 \mid a_1 * a_2 \in AExp \\ b &::= t \mid a_1 = a_2 \mid a_1 > a_2 \mid \neg b \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \in BExp \\ c &::= [\text{skip}]' \mid [x := a]' \mid c_1 ; c_2 \mid \\ &\quad \text{if } [b]' \text{ then } c_1 \text{ else } c_2 \mid \text{while } [b]' \text{ do } c \mid [\text{assert } b]' \in Cmd \end{aligned}$$

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To be done:

- Definition of **transfer functions** for **assert** blocks (depending on analysis problem)
- Idea: assertions as **filters** that let only “valid” information pass

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Constant Propagation Analysis with Assertions I

So far:

- Complete lattice $(D, \sqsubseteq)$ where
 - $D := \{\delta \mid \delta : \text{Var}_c \rightarrow \mathbb{Z} \cup \{\perp, \top\}\}$
 - $\delta(x) = z \in \mathbb{Z}$: x has **constant value** z
 - $\delta(x) = \perp$: x **undefined**
 - $\delta(x) = \top$: x **overdefined** (i.e., different possible values)
 - $\sqsubseteq \subseteq D \times D$ defined by pointwise extension of $\perp \sqsubseteq z \sqsubseteq \top$ (for every $z \in \mathbb{Z}$)
- Transfer functions $\{\varphi_l \mid l \in L\}$ defined by

$$\varphi_l(\delta) := \begin{cases} \delta & \text{if } B^l = \text{skip} \text{ or } B^l \in \text{BExp} \\ \delta[x \mapsto \text{val}_\delta(a)] & \text{if } B^l = (x := a) \end{cases}$$

where

$$\begin{aligned} \text{val}_\delta(x) &:= \delta(x) \\ \text{val}_\delta(z) &:= z \end{aligned} \quad \text{val}_\delta(a_1 \text{ op } a_2) := \begin{cases} z_1 \text{ op } z_2 & \text{if } z_1, z_2 \in \mathbb{Z} \\ \perp & \text{if } z_1 = \perp \text{ or } z_2 = \perp \\ \top & \text{otherwise} \end{cases}$$

for $z_1 := \text{val}_\delta(a_1)$ and $z_2 := \text{val}_\delta(a_2)$

Constant Propagation Analysis with Assertions II

Additionally for $B' = (\text{assert } b)$, $\delta : \text{Var}_c \rightarrow \mathbb{Z} \cup \{\perp, \top\}$ and $x \in \text{Var}_c$:

$$\varphi_I(\delta)(x) := \begin{cases} \perp & \text{if } \nexists \sigma \in \Sigma_\delta : \text{val}_\sigma(b) = \text{true} \\ z & \text{if } \forall \sigma \in \Sigma_\delta : \text{val}_\sigma(b) = \text{true} \implies \sigma(x) = z \\ \top & \text{otherwise} \end{cases}$$

where

- the **set of δ -assignments** is given by

$$\Sigma_\delta := \left\{ \sigma : \text{Var}_c \rightarrow \mathbb{Z} \mid \forall y \in \text{Var}_c : \sigma(y) \in \begin{cases} \emptyset & \text{if } \delta(y) = \perp \\ \{z\} & \text{if } \delta(y) = z \\ \mathbb{Z} & \text{if } \delta(y) = \top \end{cases} \right\}$$

(and thus $\Sigma_\delta = \emptyset$ iff $\delta(y) = \perp$ for some $y \in \text{Var}_c$)

- the **evaluation function** $\text{val}_\sigma : \text{BExp} \rightarrow \mathbb{B}$ is defined by

$$\begin{aligned} \text{val}_\sigma(t) &:= t \\ \text{val}_\sigma(a_1 = a_2) &:= (\text{val}_\sigma(a_1) = \text{val}_\sigma(a_2)) \\ \text{val}_\sigma(\neg b) &:= \begin{cases} \text{true} & \text{if } \text{val}_\sigma(b) = \text{false} \\ \text{false} & \text{otherwise} \end{cases} \\ \text{val}_\sigma(b_1 \wedge b_2) &:= \begin{cases} \text{true} & \text{if } \text{val}_\sigma(b_1) = \text{val}_\sigma(b_2) = \text{true} \\ \text{false} & \text{otherwise} \end{cases} \end{aligned}$$

etc.

Example 9.7

$$\textcircled{1} \text{ } Var_c = \{x, y, z\}, \delta = (\underbrace{\perp}_x, \underbrace{1}_y, \underbrace{\top}_z)$$

$$\implies \Sigma_\delta = \emptyset$$

$$\implies \varphi_{\text{assert } b}(\delta) = (\perp, \perp, \perp) \text{ for every } b \in BExp$$

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$$\textcircled{2} \text{ } Var_c = \{x, y, z\}, \delta = (\underbrace{1}_x, \underbrace{2}_y, \underbrace{\top}_z)$$

$$\implies \Sigma_\delta = \{(1, 2, z) \mid z \in \mathbb{Z}\}$$

$$\implies \varphi_{\text{assert } x=y}(\delta) = (\perp, \perp, \perp)$$

$$\varphi_{\text{assert } y=z}(\delta) = (1, 2, 2)$$

$$\varphi_{\text{assert } y < z}(\delta) = (1, 2, \top)$$

$$\varphi_{\text{assert } x \leq z \wedge y > z}(\delta) = (1, 2, 1)$$

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Remarks:

- Note that for $B^l = (\text{assert } b)$ and $\delta : Var_c \rightarrow \mathbb{Z} \cup \{\perp, \top\}$, $\varphi_l(\delta) \sqsubseteq \delta$ and hence $\Sigma_{\varphi_l(\delta)} \subseteq \Sigma_\delta$ (“filter”)
- If $CP_l(x) = \perp$ for some $l \in L_c$ and $x \in Var_c$, then l is **unreachable**

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Interval Analysis with Assertions I

So far:

- The domain $(Int, \subseteq)$ of **intervals over $\mathbb{Z}$** is defined by
$$Int := \{[z_1, z_2] \mid z_1 \in \mathbb{Z} \cup \{-\infty\}, z_2 \in \mathbb{Z} \cup \{+\infty\}, z_1 \leq z_2\} \cup \{\emptyset\}$$

where

- $-\infty \leq z, z \leq +\infty$, and $-\infty \leq +\infty$ (for all $z \in \mathbb{Z}$)
- $\emptyset \subseteq I$ (for all $I \in Int$)
- $[y_1, y_2] \subseteq [z_1, z_2]$ iff $z_1 \leq y_1$ and $y_2 \leq z_2$
- Transfer functions** $\{\varphi_I \mid I \in L\}$ are defined by
$$\varphi_I(\delta) := \begin{cases} \delta & \text{if } B^I = \text{skip or } B^I \in BExp \\ \delta[x \mapsto val_\delta(a)] & \text{if } B^I = (x := a) \end{cases}$$

where

$$\begin{array}{ll} val_\delta(x) := \delta(x) & val_\delta(a_1 + a_2) := val_\delta(a_1) \oplus val_\delta(a_2) \\ val_\delta(z) := [z, z] & val_\delta(a_1 - a_2) := val_\delta(a_1) \ominus val_\delta(a_2) \\ & val_\delta(a_1 * a_2) := val_\delta(a_1) \odot val_\delta(a_2) \end{array}$$

with

$$\begin{aligned} \emptyset \oplus I &:= I \oplus \emptyset := \emptyset \ominus I := \dots := \emptyset \\ [y_1, y_2] \oplus [z_1, z_2] &:= [y_1 + z_1, y_2 + z_2] \\ [y_1, y_2] \ominus [z_1, z_2] &:= [y_1 - z_2, y_2 - z_1] \\ [y_1, y_2] \odot [z_1, z_2] &:= [\min_{y \in [y_1, y_2], z \in [z_1, z_2]} y \cdot z, \max_{y \in [y_1, y_2], z \in [z_1, z_2]} y \cdot z] \end{aligned}$$

Additionally for $B^I = (\text{assert } b)$, $\delta : \text{Var}_c \rightarrow \mathbb{Z} \cup \{\perp, \top\}$ and $x \in \text{Var}_c$:

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where

- $Z := \{\sigma(x) \mid \sigma \in \Sigma_\delta, \text{val}_\sigma(b) = \text{true}\}$
- $\Sigma_\delta := \{\sigma : \text{Var}_c \rightarrow \mathbb{Z} \mid \forall y \in \text{Var}_c : \sigma(y) \in \delta(y)\}$
(and thus $\Sigma_\delta = \emptyset$ iff $\delta(y) = \emptyset$ for some $y \in \text{Var}_c$)
- $\text{val}_\sigma : B\text{Exp} \rightarrow \mathbb{B}$ as before

Example 9.8

$$Var_c = \{\mathbf{x}, \mathbf{y}\}, \delta = (\underbrace{[-\infty, 2]}_{\mathbf{x}}, \underbrace{[0, +\infty]}_{\mathbf{y}})$$

$$\Rightarrow \varphi_{\text{assert } \mathbf{x} > 0}(\delta) = ([1, 2], [0, +\infty])$$

$$\varphi_{\text{assert } \mathbf{x} = \mathbf{y}}(\delta) = ([0, 2], [0, 2])$$

$$\varphi_{\text{assert } \mathbf{x} > \mathbf{y}}(\delta) = ([1, 2], [0, 1])$$

$$\varphi_{\text{assert } \mathbf{x} < \mathbf{y}}(\delta) = ([-\infty, 2], [0, +\infty])$$

Example 9.8

$$Var_c = \{x, y\}, \delta = (\underbrace{[-\infty, 2]}_x, \underbrace{[0, +\infty]}_y)$$

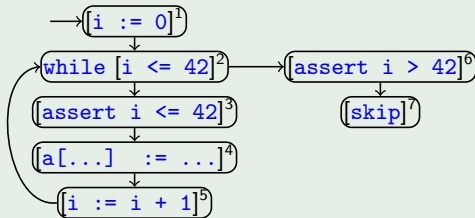
$$\begin{aligned}\Rightarrow \varphi_{\text{assert } x>0}(\delta) &= ([1, 2], [0, +\infty]) \\ \varphi_{\text{assert } x=y}(\delta) &= ([0, 2], [0, 2]) \\ \varphi_{\text{assert } x>y}(\delta) &= ([1, 2], [0, 1]) \\ \varphi_{\text{assert } x<y}(\delta) &= ([-\infty, 2], [0, +\infty])\end{aligned}$$

Remarks:

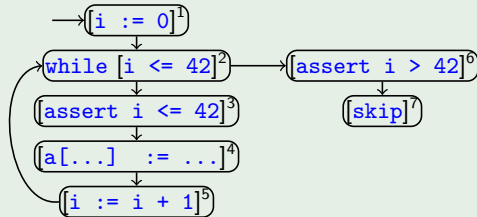
- Again for $B^I = (\text{assert } b)$ and $\delta : Var_c \rightarrow \mathbb{Z} \cup \{\perp, \top\}$, $\varphi_I(\delta) \sqsubseteq \delta$ and hence $\Sigma_{\varphi_I(\delta)} \subseteq \Sigma_\delta$ (“filter”)
- Again if $Al_I(x) = \emptyset$ for some $I \in L_c$ and $x \in Var_c$, then I is unreachable

Interval Analysis with Assertions IV

Example 9.9 (Interval analysis for `i`; cf. Example 7.7)



Example 9.9 (Interval analysis for i ; cf. Example 7.7)



$$\varphi_1(I) = [0, 0]$$

$$\varphi_2(I) = I$$

$$\varphi_3(I) = I \cap [-\infty, 42]$$

$$\varphi_4(I) = I$$

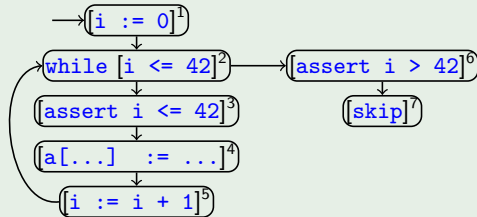
$$\varphi_5(\emptyset) = \emptyset$$

$$\varphi_5([i_1, i_2]) = [i_1 + 1, i_2 + 1]$$

$$\varphi_6(I) = I \cap [43, +\infty]$$

Interval Analysis with Assertions IV

Example 9.9 (Interval analysis for i ; cf. Example 7.7)

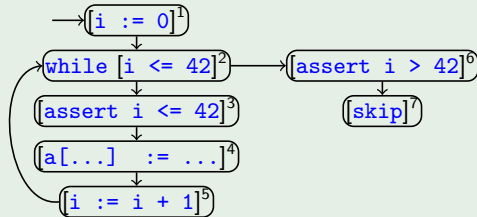


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 \varphi_6(I) &= I \cap [43, +\infty]
 \end{aligned}$$

W	AI_1	AI_2	AI_3	AI_4	AI_5	AI_6	AI_7
12, 23, 34, 45, 52, 26, 67	$[-\infty, +\infty]$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Interval Analysis with Assertions IV

Example 9.9 (Interval analysis for i ; cf. Example 7.7)

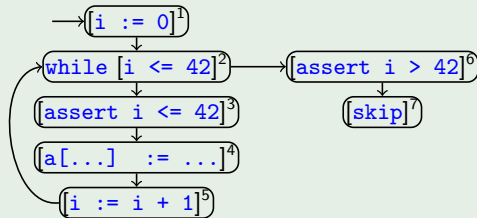


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Interval Analysis with Assertions IV

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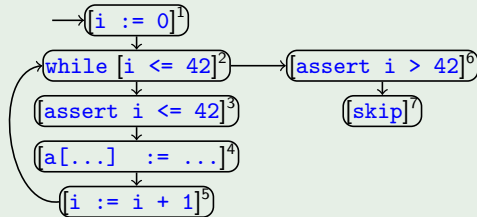


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34, 45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

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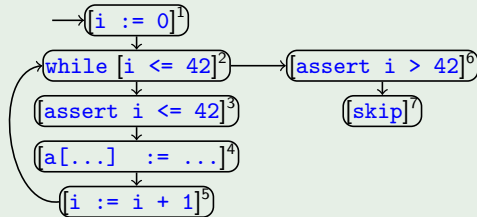
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34, 45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$

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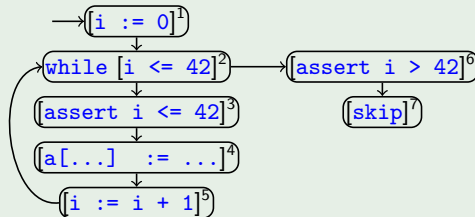
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23, 34, 45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
34, 45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$
52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$

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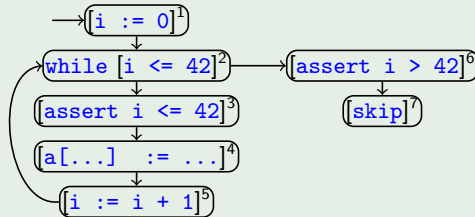
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34, 45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$
52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$
23, 26, 67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$

Interval Analysis with Assertions IV

Example 9.9 (Interval analysis for i ; cf. Example 7.7)

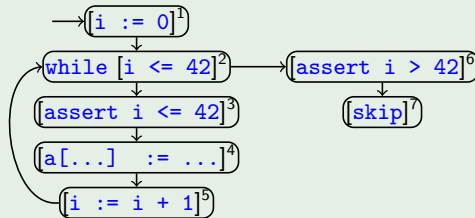


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23, 34, 45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
34, 45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$
52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$
23, 26, 67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$
34, 26, 67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, +\infty]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$

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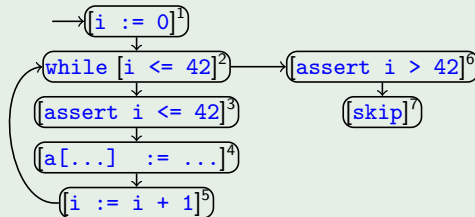
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34, 45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$
52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$
23, 26, 67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$
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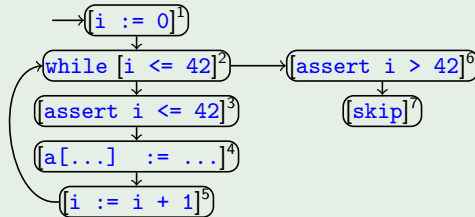
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34, 45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$
52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$
23, 26, 67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$
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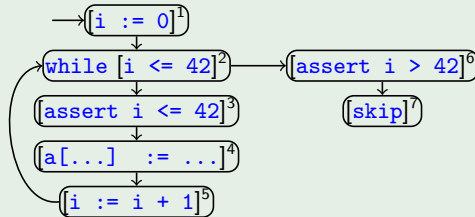
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52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$
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52, 26, 67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, +\infty]$	$[0, 42]$	$[0, 42]$	$\emptyset$	$\emptyset$
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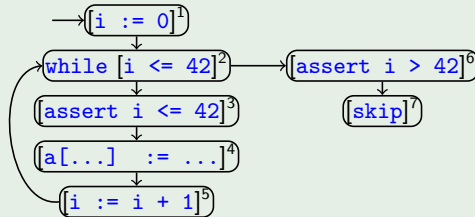


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34, 45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$
52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$
23, 26, 67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$
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45, 26, 67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, +\infty]$	$[0, 42]$	$[0, 0]$	$\emptyset$	$\emptyset$
52, 26, 67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, +\infty]$	$[0, 42]$	$[0, 42]$	$\emptyset$	$\emptyset$
26, 67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, +\infty]$	$[0, 42]$	$[0, 42]$	$\emptyset$	$\emptyset$
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34, 45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
45, 52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$	$\emptyset$
52, 26, 67	$[-\infty, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$
23, 26, 67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$
34, 26, 67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, +\infty]$	$[0, 0]$	$[0, 0]$	$\emptyset$	$\emptyset$
45, 26, 67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, +\infty]$	$[0, 42]$	$[0, 0]$	$\emptyset$	$\emptyset$
52, 26, 67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, +\infty]$	$[0, 42]$	$[0, 42]$	$\emptyset$	$\emptyset$
26, 67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, +\infty]$	$[0, 42]$	$[0, 42]$	$\emptyset$	$\emptyset$
67	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, +\infty]$	$[0, 42]$	$[0, 42]$	$[0, +\infty]$	$\emptyset$
ϵ	$[-\infty, +\infty]$	$[0, +\infty]$	$[0, +\infty]$	$[0, 42]$	$[0, 42]$	$[0, +\infty]$	$[43, +\infty]$