Semantics and Verification of Software

Lecture 15: Dataflow Analysis

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Outline

Labelled Programs

- Goal: localization of analysis information
- Dataflow information will be associated with
  - assignments
  - tests in conditionals (`if`) and loops (`while`)
  - `skip` statements

  These constructs will be called blocks.
- Assume set of labels Lab with meta variable $l \in Lab$ (usually $Lab = \mathbb{N}$)

Definition (Labelled WHILE programs)

The syntax of labelled WHILE programs is defined by the following context–free grammar:

$$\begin{aligned} a &::= z \mid x \mid a_1\texttt{+}a_2 \mid a_1\texttt{-}a_2 \mid a_1\texttt{*}a_2 \in AExp \\ b &::= t \mid a_1\texttt{=}a_2 \mid a_1\texttt{>}a_2 \mid \neg b \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \in BExp \\ c &::= [\texttt{skip}]^l \mid [x \texttt{ := } a]^l \mid c_1\texttt{;}c_2 \mid \\ &\qquad \texttt{if } [b]^l \texttt{ then } c_1 \texttt{ else } c_2 \mid \texttt{while } [b]^l \texttt{ do } c \in Cmd \end{aligned}$$

Here all labels in a statement $c \in Cmd$ are assumed to be distinct.

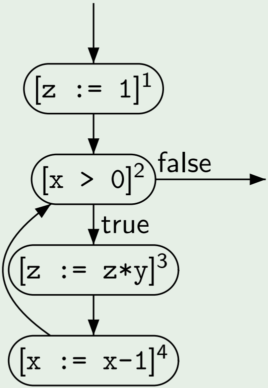
Representing Control Flow

Example

$c =$ [z := 1]^1;
while [x > 0]^2 do
[z := z*y]^3;
[x := x-1]^4

$\mathsf{init}(c) = 1$
$\mathsf{final}(c) = \{2\}$
$\mathsf{flow}(c) = \{(1,2),(2,3),(3,4),(4,2)\}$

Visualization by flow graph:



Goal of Available Expressions Analysis

Available Expressions Analysis

The goal of Available Expressions Analysis is to determine, for each program point, which (complex) expressions *must* have been computed, and not later modified, on all paths to the program point.

- can be used to avoid recomputations of expressions
- for simplicity: only non–trivial arithmetic expressions

Example (Available Expressions analysis)

```
[x := a+b]^1;
[y := a*b]^2;
while [y > a+b]^3 do
  [a := a+1]^4;
  [x := a+b]^5
```

- `a+b` available at label 3
- `a+b` not available at label 5
- possible optimization: `while [y > x]^3 do`

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Formalizing Available Expressions Analysis I

- Given $c \in Cmd$, $Lab_c / Block_c / AExp_c$ denote the sets of all labels/blocks/complex arithmetic expressions occurring in c, respectively
- An expression a is killed in a block B if any of the variables in a is modified in B
- Formally: $\mathsf{kill}_{\mathsf{AE}} : Block_c \to 2^{AExp_c}$ is defined by

$$\begin{aligned} \mathsf{kill}_{\mathsf{AE}}([\mathtt{skip}]^l) &:= \emptyset \\ \mathsf{kill}_{\mathsf{AE}}([x \mathtt{\ :=\ } a]^l) &:= \{a' \in AExp_c \mid x \in FV(a')\} \\ \mathsf{kill}_{\mathsf{AE}}([b]^l) &:= \emptyset \end{aligned}$$

- An expression a is generated in a block B if it is evaluated in and none of its variables are modified by B
- Formally: $\mathsf{gen}_{\mathsf{AE}} : Block_c \to 2^{AExp_c}$ is defined by

$$\begin{aligned} \mathsf{gen}_{\mathsf{AE}}([\mathtt{skip}]^l) &:= \emptyset \\ \mathsf{gen}_{\mathsf{AE}}([x \mathtt{\ :=\ } a]^l) &:= \{a \mid x \notin FV(a)\} \\ \mathsf{gen}_{\mathsf{AE}}([b]^l) &:= AExp_b \end{aligned}$$

Formalizing Available Expressions Analysis II

Example ($\mathsf{kill}_{\mathsf{AE}}/\mathsf{gen}_{\mathsf{AE}}$ functions)

$c =$ `[x := a+b]`^{1}`;`
`[y := a*b]`^{2}`;`
`while [y > a+b]`^{3} `do`
`[a := a+1]`^{4}`;`
`[x := a+b]`^{5}

- $AExp_c = \{\mathtt{a+b}, \mathtt{a*b}, \mathtt{a+1}\}$
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4	{a+b, a*b, a+1}	$\emptyset$
5	$\emptyset$	{a+b}

Outline

The Equation System I

- Analysis itself defined by setting up an equation system
- For each $l \in Lab_c$, $\mathsf{AE}_l \subseteq AExp_c$ represents the set of available expressions at the entry of block B^l
- Formally, for $c \in Cmd$ with isolated entry:

$$\mathsf{AE}_l = \begin{cases} \emptyset & \text{if } l = \mathsf{init}(c) \\ \bigcap\{\varphi_{l'}(\mathsf{AE}_{l'}) \mid (l', l) \in \mathsf{flow}(c)\} & \text{otherwise} \end{cases}$$

  where $\varphi_{l'} : 2^{AExp_c} \to 2^{AExp_c}$ denotes the transfer function of block $B^{l'}$, given by

$$\varphi_{l'}(A) := (A \setminus \mathsf{kill}_{\mathsf{AE}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{AE}}(B^{l'})$$

- Characterization of analysis:

  forward: starts in $\mathsf{init}(c)$ and proceeds downwards
  must: $\bigcap$ in equation for AE_l
  flow–sensitive: results depending on order of assignments

- Later: solution not necessarily unique
  $\Longrightarrow$ choose greatest one

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Reminder:

$$\mathsf{AE}_l = \begin{cases} \emptyset & \text{if } l = \mathsf{init}(c) \\ \bigcap\{\varphi_{l'}(\mathsf{AE}_{l'}) \mid (l',l) \in \mathsf{flow}(c)\} & \text{otherwise} \end{cases}$$

$$\varphi_{l'}(E) = (E \setminus \mathsf{kill}_{\mathsf{AE}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{AE}}(B^{l'})$$

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$c = [\texttt{x := a+b}]^1;$
$\quad [\texttt{y := a*b}]^2;$
$\quad \texttt{while } [\texttt{y > a+b}]^3 \texttt{ do}$
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Equations:

$$\begin{aligned}
\mathsf{AE}_1 &= \emptyset \\
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\mathsf{AE}_3 &= \varphi_2(\mathsf{AE}_2) \cap \varphi_5(\mathsf{AE}_5) \\
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Goal of the Analysis

Live Variables Analysis

The goal of Live Variables Analysis is to determine, for each program point, which variables *may* be live at the exit from the point.

- A variable is called live at the exit from a block if there exists a path from the block to a use of the variable that does not re–define the variable
- No variable considered to be live at the end of the program (choice depends on application, e.g., output variables live)
- Can be used for Dead Code Elimination: remove assignments to non–live variables

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An Example

Example 15.2 (Live Variables Analysis)

```
[x := 2]^1;
[y := 4]^2;
[x := 1]^3;
if [y > 0]^4 then
  [z := x]^5
else
  [z := y*y]^6;
[x := z]^7
```

- x not live at exit from label 1
- y live at exit from 2
- x live at exit from 3
- z live at exits from 5 and 6
- possible optimization: remove [x := 2]^1

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Formalizing Live Variables Analysis I

- A variable on the left–hand side of an assignment is killed by the assignment; tests and `skip` do not kill
- Formally: $\mathsf{kill_{LV}} : Block_c \to 2^{Var_c}$ is defined by

$$\begin{aligned}\mathsf{kill_{LV}}([\mathtt{skip}]^l) &:= \emptyset\\ \mathsf{kill_{LV}}([x := a]^l) &:= \{x\}\\ \mathsf{kill_{LV}}([b]^l) &:= \emptyset\end{aligned}$$

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$c =$ [x := 2]^1;
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- $Var_c = \{\mathrm{x}, \mathrm{y}, \mathrm{z}\}$
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The Equation System I

- For each $l \in Lab_c$, $\mathsf{LV}_l \subseteq Var_c$ represents the set of live variables at the exit of block B^l
- Formally, for a program $c \in Cmd$ with isolated exits:

$$\mathsf{LV}_l = \begin{cases} \emptyset & \text{if } l \in \mathsf{final}(c) \\ \bigcup\{\varphi_{l'}(\mathsf{LV}_{l'}) \mid (l,l') \in \mathsf{flow}(c)\} & \text{otherwise} \end{cases}$$

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$$\varphi_{l'}(V) := (V \setminus \mathsf{kill}_{\mathsf{LV}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{LV}}(B^{l'})$$

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  may: $\bigcup$ in equation for LV_l

  flow–sensitive: results depending on order of assignments

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The Equation System II

Reminder:

$$\mathsf{LV}_l = \begin{cases} \emptyset & \text{if } l \in \mathsf{final}(c) \\ \bigcup\{\varphi_{l'}(\mathsf{LV}_{l'}) \mid (l,l') \in \mathsf{flow}(c)\} & \text{otherwise} \end{cases}$$

$$\varphi_{l'}(V) = (V \setminus \mathsf{kill}_{\mathsf{AE}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{AE}}(B^{l'})$$

Example 15.4 (LV equation system)

$c =$
```
[x := 2]^1;[y := 4]^2;
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$$\begin{aligned}
\mathsf{LV}_1 &= \varphi_2(\mathsf{LV}_2) = \mathsf{LV}_2 \setminus \{y\} \\
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Solution:
$$\begin{aligned}
\mathsf{LV}_1 &= \emptyset \\
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\mathsf{LV}_3 &= \{x,y\} \\
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Outline

Similarities between Analysis Problems

- **Observation:** the analyses presented so far have some similarities

$\Longrightarrow$ Look for underlying framework

- **Advantage:** possibility for designing (efficient) generic algorithms for solving dataflow equations
- **Overall pattern:** for $c \in Cmd$ and $l \in Lab_c$, the analysis information (AI) is described by equations of the form

$$\mathsf{AI}_l = \begin{cases} \iota & \text{if } l \in E \\ \bigsqcup\{\varphi_{l'}(\mathsf{AI}_{l'}) \mid (l', l) \in F\} & \text{otherwise} \end{cases}$$

where

- ι specifies the initial analysis information
- E is $\{\mathsf{init}(c)\}$ or $\mathsf{final}(c)$
- $\bigsqcup$ is $\bigcap$ or $\bigcup$
- $\varphi_{l'}$ denotes the transfer function of block $B^{l'}$
- F is $\mathsf{flow}(c)$ or $\mathsf{flow}^R(c)$ $(:= \{(l', l) \mid (l, l') \in \mathsf{flow}(c)\})$

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Characterization of Analyses

- **Direction of information flow:**
  - forward:
    - $F = \mathsf{flow}(c)$
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    - c has isolated entry
  - backward:
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- **Quantification over paths:**
  - may:
    - $\bigsqcup = \bigcup$
    - property satisfied by some path
    - interested in least solution (later)
  - must:
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