Semantics and Verification of Software

Lecture 17: Dataflow Analysis

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Outline

Complete Lattices

Definition (Complete lattices)

A complete lattice is a partial order $(D, \sqsubseteq)$ such that all subsets of D have least upper as well as greatest lower bounds. In this case,

$$\bot := \bigsqcup \emptyset = \bigsqcap D \text{ and}$$
$$\top := \bigsqcap \emptyset = \bigsqcup D$$

denote the least and the greatest element of D, respectively.

Example

1. (Available Expressions) $(D, \sqsubseteq) = (2^{AExp_c}, \supseteq)$ is a complete lattice with $\bot = AExp_c$ and $\top = \emptyset$
2. (Live Variables) $(D, \sqsubseteq) = (2^{Var_c}, \subseteq)$ is a complete lattice with $\bot = \emptyset$ and $\top = Var_c$

Chains

Chains represent the approximation of the analysis information.

Definition (Chain; repetition of Def. 5.6 and 5.8)

Let $(D, \sqsubseteq)$ be a partial order.

1. A subset $S \subseteq D$ is called a chain in D if, for every $s_1, s_2 \in S$,
$$s_1 \sqsubseteq s_2 \text{ or } s_2 \sqsubseteq s_1$$
(that is, S is a totally ordered subset of D).
2. $(D, \sqsubseteq)$ is called chain complete (CCPO) if every of its chains has a least upper bound.
3. $(D, \sqsubseteq)$ satisfies the Ascending Chain Condition (ACC) if each ascending chain $d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ eventually stabilizes, i.e., there exists $n \in \mathbb{N}$ such that $d_n = d_{n+1} = \ldots$

Monotonicity of Functions

Definition (Monotonicity; repetition of Def. 6.1)

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be partial orders, and let $F : D \to D'$. F is called monotonic (w.r.t. $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$) if, for every $d_1, d_2 \in D$,

$$d_1 \sqsubseteq d_2 \implies F(d_1) \sqsubseteq' F(d_2).$$

Example

1. (Available Expressions) $(D, \sqsubseteq) = (2^{AExp_c}, \supseteq)$
   Each transfer function $\varphi_{l'}(A) := (A \setminus \mathsf{kill}_{\mathsf{AE}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{AE}}(B^{l'})$ is monotonic
2. (Live Variables) $(D, \sqsubseteq) = (2^{Var_c}, \subseteq)$
   Each transfer function $\varphi_{l'}(V) := (V \setminus \mathsf{kill}_{\mathsf{LV}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{LV}}(B^{l'})$ is monotonic

Fixpoints

Theorem (Fixpoint Theorem; repetition of Thm. 7.1)

Let $(D, \sqsubseteq)$ be a CCPO and $F : D \to D$ continuous. Then

$$\mathsf{fix}(F) := \bigsqcup \{F^n(\bigsqcup \emptyset) \mid n \in \mathbb{N}\}$$

is the least fixpoint of F.

Definition (Continuity; repetition of Def. 6.5)

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be CCPOs and $F : D \to D'$ monotonic. Then F is called continuous (w.r.t. $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$) if, for every non–empty chain $S \subseteq D$,

$$F(\bigsqcup S) = \bigsqcup F(S).$$

Corollary

Montonic functions on partial orders that satisfy ACC are continuous.

Dataflow Systems I

Definition (Dataflow system)

A dataflow system $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$ consists of

- a finite set of (program) labels Lab (here: Lab_c),
- a set of extremal labels $E \subseteq Lab$ (here: $\{\mathsf{init}(c)\}$ or $\mathsf{final}(c)$),
- a flow relation $F \subseteq Lab \times Lab$ (here: $\mathsf{flow}(c)$ or $\mathsf{flow}^R(c)$),
- a complete lattice $(D, \sqsubseteq)$ that satisfies ACC (with LUB operator $\bigsqcup$ and least element $\bot$),
- an extremal value $\iota \in D$ (for the extremal labels), and
- a collection of monotonic transfer functions $\{\varphi_l \mid l \in Lab\}$ of type $\varphi_l : D \to D$.

Example

Problem	**Available Expressions**	**Live Variables**
E	$\{\mathsf{init}(c)\}$	$\mathsf{final}(c)$
F	$\mathsf{flow}(c)$	$\mathsf{flow}^R(c)$
D	2^{AExp_c}	2^{Var_c}
$\sqsubseteq$	$\supseteq$	$\subseteq$
$\bigsqcup$	$\bigcap$	$\bigcup$
$\bot$	$AExp_c$	$\emptyset$
ι	$\emptyset$	$\emptyset$
φ_l	$\varphi_l(d) = (d \setminus \mathsf{kill}(B^l)) \cup \mathsf{gen}(B^l)$	

Outline

The Equation System

Definition 17.1 (Dataflow equation system)

Let $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$ be a dataflow system. S defines the following equation system over the set of variables $\{\mathsf{AI}_l \mid l \in Lab\}$:

$$\mathsf{AI}_l = \begin{cases} \iota & \text{if } l \in E \\ \bigsqcup\{\varphi_{l'}(\mathsf{AI}_{l'}) \mid (l', l) \in F\} & \text{otherwise} \end{cases}$$

The Functional

Just as in the denotational semantics of `while` loops, the equation system determines a functional whose fixpoints are the solutions of the equation system.

Definition 17.2 (Dataflow functional)

The equation system of a dataflow system $S = (Lab, E, F, (D, \sqsubseteq), \iota, \varphi)$ induces a functional

$$\Phi_S : D^n \to D^n : (d_{l_1}, \ldots, d_{l_n}) \mapsto (d'_{l_1}, \ldots, d'_{l_n})$$

where $Lab = \{l_1, \ldots, l_n\}$ and

$$d'_{l_i} := \begin{cases} \iota & \text{if } l \in E \\ \bigsqcup\{\varphi_{l'}(d_{l'}) \mid (l', l_i) \in F\} & \text{otherwise} \end{cases}$$

Fixpoint Iteration I

Remarks:

- $(d_1, \ldots, d_n)$ is a solution of the equation system iff it is a fixpoint of Φ_S
- If $(D, \sqsubseteq)$ is a complete lattice satisfying ACC, then so is $(D^n, \sqsubseteq^n)$ (where $(d_1, \ldots, d_n) \sqsubseteq^n (d'_1, \ldots, d'_n)$ iff $d_i \sqsubseteq d'_i$ for every $1 \leq i \leq n$)
- Every transfer function φ_l monotonic in D
  $\Longrightarrow$ Φ_S monotonic in D^n
- Thus the fixpoint is effectively computable by iteration:

$$\mathsf{fix}(\Phi_S) = \bigsqcup\{\Phi_S^i(\bot_{D^n}) \mid i \in \mathbb{N}\}$$

  where $\bot_{D^n} = \underbrace{(\bot_D, \ldots, \bot_D)}_{n \text{ times}}$

- If maximal length of chains in D is m
  $\Longrightarrow$ maximal length of chains in D^n is $m \cdot n$
  $\Longrightarrow$ fixpoint iteration requires at most $m \cdot n$ steps

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- Every transfer function φ_l monotonic in D
  $\implies \Phi_S$ monotonic in D^n
- Thus the fixpoint is effectively computable by iteration:

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  $\Longrightarrow$ maximal length of chains in D^n is $m \cdot n$
  $\Longrightarrow$ fixpoint iteration requires at most $m \cdot n$ steps

Example 17.3 (Available Expressions; cf. Example 15.1)

Program:

$c = $
```
[x := a+b]^1;
[y := a*b]^2;
while [y > a+b]^3 do
  [a := a+1]^4;
  [x := a+b]^5
```

Equation system:

$$\mathsf{AE}_1 = \emptyset$$
$$\mathsf{AE}_2 = \mathsf{AE}_1 \cup \{\mathtt{a+b}\}$$
$$\mathsf{AE}_3 = (\mathsf{AE}_2 \cup \{\mathtt{a*b}\}) \cap (\mathsf{AE}_5 \cup \{\mathtt{a+b}\})$$
$$\mathsf{AE}_4 = \mathsf{AE}_3 \cup \{\mathtt{a+b}\}$$
$$\mathsf{AE}_5 = \mathsf{AE}_4 \setminus \{\mathtt{a+b}, \mathtt{a*b}, \mathtt{a+1}\}$$

Fixpoint iteration:

i	1	2	3	4	5
0	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$
1	$\emptyset$	$AExp_c$	$AExp_c$	$AExp_c$	$\emptyset$
2	$\emptyset$	{a+b}	{a+b}	$AExp_c$	$\emptyset$
3	$\emptyset$	{a+b}	{a+b}	{a+b}	$\emptyset$
4	$\emptyset$	{a+b}	{a+b}	{a+b}	$\emptyset$

Fixpoint Iteration II

Example 17.3 (Available Expressions; cf. Example 15.1)

Program:

$c =$
```
[x := a+b]^1;
[y := a*b]^2;
while [y > a+b]^3 do
  [a := a+1]^4;
  [x := a+b]^5
```

Equation system:

$$\begin{aligned}
\mathsf{AE}_1 &= \emptyset \\
\mathsf{AE}_2 &= \mathsf{AE}_1 \cup \{\mathtt{a+b}\} \\
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\mathsf{AE}_4 &= \mathsf{AE}_3 \cup \{\mathtt{a+b}\} \\
\mathsf{AE}_5 &= \mathsf{AE}_4 \setminus \{\mathtt{a+b}, \mathtt{a*b}, \mathtt{a+1}\}
\end{aligned}$$

Fixpoint iteration:

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0	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$
1	$\emptyset$	$AExp_c$	$AExp_c$	$AExp_c$	$\emptyset$
2	$\emptyset$	{a+b}	{a+b}	$AExp_c$	$\emptyset$
3	$\emptyset$	{a+b}	{a+b}	{a+b}	$\emptyset$
4	$\emptyset$	{a+b}	{a+b}	{a+b}	$\emptyset$

Fixpoint Iteration II

Example 17.3 (Available Expressions; cf. Example 15.1)

Program:

$$
\begin{aligned}
c = {} & [\texttt{x := a+b}]^1;\\
& [\texttt{y := a*b}]^2;\\
& \texttt{while } [\texttt{y > a+b}]^3 \texttt{ do}\\
& \quad [\texttt{a := a+1}]^4;\\
& \quad [\texttt{x := a+b}]^5
\end{aligned}
$$

Equation system:

$$
\begin{aligned}
\mathsf{AE}_1 &= \emptyset\\
\mathsf{AE}_2 &= \mathsf{AE}_1 \cup \{\texttt{a+b}\}\\
\mathsf{AE}_3 &= (\mathsf{AE}_2 \cup \{\texttt{a*b}\}) \cap (\mathsf{AE}_5 \cup \{\texttt{a+b}\})\\
\mathsf{AE}_4 &= \mathsf{AE}_3 \cup \{\texttt{a+b}\}\\
\mathsf{AE}_5 &= \mathsf{AE}_4 \setminus \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\}
\end{aligned}
$$

Fixpoint iteration:

i	1	2	3	4	5
0	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$
1	$\emptyset$	$AExp_c$	$AExp_c$	$AExp_c$	$\emptyset$
2	$\emptyset$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$AExp_c$	$\emptyset$
3	$\emptyset$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$\emptyset$
4	$\emptyset$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$\emptyset$

Fixpoint Iteration II

Example 17.3 (Available Expressions; cf. Example 15.1)

Program:

$c = $ [x := a+b]^1;
  [y := a*b]^2;
  while [y > a+b]^3 do
    [a := a+1]^4;
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\mathsf{AE}_4 &= \mathsf{AE}_3 \cup \{\texttt{a+b}\} \\
\mathsf{AE}_5 &= \mathsf{AE}_4 \setminus \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\}
\end{aligned}$$

Fixpoint iteration:

i	1	2	3	4	5
0	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$
1	$\emptyset$	$AExp_c$	$AExp_c$	$AExp_c$	$\emptyset$
2	$\emptyset$	{a+b}	{a+b}	$AExp_c$	$\emptyset$
3	$\emptyset$	{a+b}	{a+b}	{a+b}	$\emptyset$
4	$\emptyset$	{a+b}	{a+b}	{a+b}	$\emptyset$

Fixpoint Iteration II

Example 17.3 (Available Expressions; cf. Example 15.1)

Program:

```
c = [x := a+b]^1;
    [y := a*b]^2;
    while [y > a+b]^3 do
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```

Equation system:

$$\begin{aligned} \mathsf{AE}_1 &= \emptyset \\ \mathsf{AE}_2 &= \mathsf{AE}_1 \cup \{\mathtt{a+b}\} \\ \mathsf{AE}_3 &= (\mathsf{AE}_2 \cup \{\mathtt{a*b}\}) \cap (\mathsf{AE}_5 \cup \{\mathtt{a+b}\}) \\ \mathsf{AE}_4 &= \mathsf{AE}_3 \cup \{\mathtt{a+b}\} \\ \mathsf{AE}_5 &= \mathsf{AE}_4 \setminus \{\mathtt{a+b}, \mathtt{a*b}, \mathtt{a+1}\} \end{aligned}$$

Fixpoint iteration:

i	1	2	3	4	5
0	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$
1	$\emptyset$	$AExp_c$	$AExp_c$	$AExp_c$	$\emptyset$
2	$\emptyset$	{a+b}	{a+b}	$AExp_c$	$\emptyset$
3	$\emptyset$	{a+b}	{a+b}	{a+b}	$\emptyset$
4	$\emptyset$	{a+b}	{a+b}	{a+b}	$\emptyset$

Fixpoint Iteration II

Example 17.3 (Available Expressions; cf. Example 15.1)

Program:

$c =$ `[x := a+b]`^{1}`;`
`[y := a*b]`^{2}`;`
`while [y > a+b]`^{3} `do`
`[a := a+1]`^{4}`;`
`[x := a+b]`^{5}

Equation system:

$$\begin{aligned}
\mathsf{AE}_1 &= \emptyset\\
\mathsf{AE}_2 &= \mathsf{AE}_1 \cup \{\texttt{a+b}\}\\
\mathsf{AE}_3 &= (\mathsf{AE}_2 \cup \{\texttt{a*b}\}) \cap (\mathsf{AE}_5 \cup \{\texttt{a+b}\})\\
\mathsf{AE}_4 &= \mathsf{AE}_3 \cup \{\texttt{a+b}\}\\
\mathsf{AE}_5 &= \mathsf{AE}_4 \setminus \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\}
\end{aligned}$$

Fixpoint iteration:

i	1	2	3	4	5
0	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$
1	$\emptyset$	$AExp_c$	$AExp_c$	$AExp_c$	$\emptyset$
2	$\emptyset$	{a+b}	{a+b}	$AExp_c$	$\emptyset$
3	$\emptyset$	{a+b}	{a+b}	{a+b}	$\emptyset$
4	$\emptyset$	{a+b}	{a+b}	{a+b}	$\emptyset$

Fixpoint Iteration II

Example 17.3 (Available Expressions; cf. Example 15.1)

Program:

$c = [\texttt{x := a+b}]^1;$
$\quad [\texttt{y := a*b}]^2;$
$\quad \texttt{while } [\texttt{y > a+b}]^3 \texttt{ do}$
$\qquad [\texttt{a := a+1}]^4;$
$\qquad [\texttt{x := a+b}]^5$

Equation system:

$$\begin{aligned}
\mathsf{AE}_1 &= \emptyset \\
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\mathsf{AE}_5 &= \mathsf{AE}_4 \setminus \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\}
\end{aligned}$$

Fixpoint iteration:

i	1	2	3	4	5
0	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$
1	$\emptyset$	$AExp_c$	$AExp_c$	$AExp_c$	$\emptyset$
2	$\emptyset$	{a+b}	{a+b}	$AExp_c$	$\emptyset$
3	$\emptyset$	{a+b}	{a+b}	{a+b}	$\emptyset$
4	$\emptyset$	{a+b}	{a+b}	{a+b}	$\emptyset$

Fixpoint Iteration III

Example 17.4 (Live Variables; cf. Example 15.4)

Program:

```
[x := 2]^1;[y := 4]^2;
[x := 1]^3;
if [y > 0]^4 then
  [z := x]^5
else
  [z := y*y]^6;
[x := z]^7
```

Equation system:

$$\mathsf{LV}_1 = \mathsf{LV}_2 \setminus \{y\}$$
$$\mathsf{LV}_2 = \mathsf{LV}_3 \setminus \{x\}$$
$$\mathsf{LV}_3 = \mathsf{LV}_4 \cup \{y\}$$
$$\mathsf{LV}_4 = ((\mathsf{LV}_5 \setminus \{z\}) \cup \{x\}) \cup ((\mathsf{LV}_6 \setminus \{z\}) \cup \{y\})$$
$$\mathsf{LV}_5 = (\mathsf{LV}_7 \setminus \{x\}) \cup \{z\}$$
$$\mathsf{LV}_6 = (\mathsf{LV}_7 \setminus \{x\}) \cup \{z\}$$
$$\mathsf{LV}_7 = \emptyset$$

Fixpoint iteration:

i	1	2	3	4	5	6	7
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{z\}$	$\{z\}$	$\emptyset$
2	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{x, y\}$	$\{z\}$	$\{z\}$	$\emptyset$
3	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{x, y\}$	$\{z\}$	$\{z\}$	$\emptyset$

Example 17.4 (Live Variables; cf. Example 15.4)

Program:

```
[x := 2]^1;[y := 4]^2;
[x := 1]^3;
if [y > 0]^4 then
  [z := x]^5
else
  [z := y*y]^6;
[x := z]^7
```

Equation system:

$$
\begin{aligned}
LV_1 &= LV_2 \setminus \{y\} \\
LV_2 &= LV_3 \setminus \{x\} \\
LV_3 &= LV_4 \cup \{y\} \\
LV_4 &= ((LV_5 \setminus \{z\}) \cup \{x\}) \cup ((LV_6 \setminus \{z\}) \cup \{y\}) \\
LV_5 &= (LV_7 \setminus \{x\}) \cup \{z\} \\
LV_6 &= (LV_7 \setminus \{x\}) \cup \{z\} \\
LV_7 &= \emptyset
\end{aligned}
$$

Fixpoint iteration:

i	1	2	3	4	5	6	7
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{z\}$	$\{z\}$	$\emptyset$
2	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{x, y\}$	$\{z\}$	$\{z\}$	$\emptyset$
3	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{x, y\}$	$\{z\}$	$\{z\}$	$\emptyset$

Fixpoint Iteration III

Example 17.4 (Live Variables; cf. Example 15.4)

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[x := 2]^1;[y := 4]^2;
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if [y > 0]^4 then
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Equation system:

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\begin{aligned}
\mathsf{LV}_1 &= \mathsf{LV}_2 \setminus \{\mathtt{y}\} \\
\mathsf{LV}_2 &= \mathsf{LV}_3 \setminus \{\mathtt{x}\} \\
\mathsf{LV}_3 &= \mathsf{LV}_4 \cup \{\mathtt{y}\} \\
\mathsf{LV}_4 &= ((\mathsf{LV}_5 \setminus \{\mathtt{z}\}) \cup \{\mathtt{x}\}) \cup ((\mathsf{LV}_6 \setminus \{\mathtt{z}\}) \cup \{\mathtt{y}\}) \\
\mathsf{LV}_5 &= (\mathsf{LV}_7 \setminus \{\mathtt{x}\}) \cup \{\mathtt{z}\} \\
\mathsf{LV}_6 &= (\mathsf{LV}_7 \setminus \{\mathtt{x}\}) \cup \{\mathtt{z}\} \\
\mathsf{LV}_7 &= \emptyset
\end{aligned}
$$

Fixpoint iteration:

i	1	2	3	4	5	6	7
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$	$\{\mathtt{y}\}$	$\{\mathtt{x},\mathtt{y}\}$	$\{\mathtt{z}\}$	$\{\mathtt{z}\}$	$\emptyset$
2	$\emptyset$	$\{\mathtt{y}\}$	$\{\mathtt{x},\ \mathtt{y}\}$	$\{\mathtt{x},\mathtt{y}\}$	$\{\mathtt{z}\}$	$\{\mathtt{z}\}$	$\emptyset$
3	$\emptyset$	$\{\mathtt{y}\}$	$\{\mathtt{x},\ \mathtt{y}\}$	$\{\mathtt{x},\mathtt{y}\}$	$\{\mathtt{z}\}$	$\{\mathtt{z}\}$	$\emptyset$

Fixpoint Iteration III

Example 17.4 (Live Variables; cf. Example 15.4)

Program:

```
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Equation system:

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\begin{aligned}
\mathsf{LV}_1 &= \mathsf{LV}_2 \setminus \{y\}\\
\mathsf{LV}_2 &= \mathsf{LV}_3 \setminus \{x\}\\
\mathsf{LV}_3 &= \mathsf{LV}_4 \cup \{y\}\\
\mathsf{LV}_4 &= ((\mathsf{LV}_5 \setminus \{z\}) \cup \{x\}) \cup ((\mathsf{LV}_6 \setminus \{z\}) \cup \{y\})\\
\mathsf{LV}_5 &= (\mathsf{LV}_7 \setminus \{x\}) \cup \{z\}\\
\mathsf{LV}_6 &= (\mathsf{LV}_7 \setminus \{x\}) \cup \{z\}\\
\mathsf{LV}_7 &= \emptyset
\end{aligned}
$$

Fixpoint iteration:

i	1	2	3	4	5	6	7
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$	$\{y\}$	$\{x,y\}$	$\{z\}$	$\{z\}$	$\emptyset$
2	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{x,y\}$	$\{z\}$	$\{z\}$	$\emptyset$
3	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{x,y\}$	$\{z\}$	$\{z\}$	$\emptyset$

Fixpoint Iteration III

Example 17.4 (Live Variables; cf. Example 15.4)

Program:

```
[x := 2]^1;[y := 4]^2;
[x := 1]^3;
if [y > 0]^4 then
  [z := x]^5
else
  [z := y*y]^6;
[x := z]^7
```

Equation system:

$$\begin{aligned}
\mathsf{LV}_1 &= \mathsf{LV}_2 \setminus \{y\}\\
\mathsf{LV}_2 &= \mathsf{LV}_3 \setminus \{x\}\\
\mathsf{LV}_3 &= \mathsf{LV}_4 \cup \{y\}\\
\mathsf{LV}_4 &= ((\mathsf{LV}_5 \setminus \{z\}) \cup \{x\}) \cup ((\mathsf{LV}_6 \setminus \{z\}) \cup \{y\})\\
\mathsf{LV}_5 &= (\mathsf{LV}_7 \setminus \{x\}) \cup \{z\}\\
\mathsf{LV}_6 &= (\mathsf{LV}_7 \setminus \{x\}) \cup \{z\}\\
\mathsf{LV}_7 &= \emptyset
\end{aligned}$$

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