

Semantics and Verification of Software

Lecture 5: Basic Fixpoint Theory

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- 1 Repetition: Denotational Semantics
- 2 Characterization of $\text{fix}(\Phi)$
- 3 Chain-Complete Partial Orders

Definition (Denotational semantics of statements)

The (denotational) semantic functional for statements,

$$\mathcal{C}[\![\cdot]\!] : Cmd \rightarrow (\Sigma \rightarrow \Sigma),$$

is given by:

$$\begin{aligned}\mathcal{C}[\![\text{skip}]\!] &:= \text{id}_{\Sigma} \\ \mathcal{C}[\![x := a]\!]\sigma &:= \sigma[x \mapsto \mathcal{A}[\![a]\!]\sigma] \\ \mathcal{C}[\![c_1; c_2]\!] &:= \mathcal{C}[\![c_2]\!] \circ \mathcal{C}[\![c_1]\!] \\ \mathcal{C}[\![\text{if } b \text{ then } c_1 \text{ else } c_2]\!] &:= \text{cond}(\mathcal{B}[\![b]\!], \mathcal{C}[\![c_1]\!], \mathcal{C}[\![c_2]\!]) \\ \mathcal{C}[\![\text{while } b \text{ do } c]\!] &:= \text{fix}(\Phi)\end{aligned}$$

where $\Phi : (\Sigma \rightarrow \Sigma) \rightarrow (\Sigma \rightarrow \Sigma) : f \mapsto \text{cond}(\mathcal{B}[\![b]\!], f \circ \mathcal{C}[\![c]\!], \text{id}_{\Sigma})$

Why Fixpoints?

- Goal: preserve **validity of equivalence**

$$\mathcal{C}[\text{while } b \text{ do } c] = \mathcal{C}[\text{if } b \text{ then } (c; \text{while } b \text{ do } c) \text{ else skip}]$$

- Using the known parts of Def. 4.8, we obtain:

$$\begin{aligned}\mathcal{C}[\text{while } b \text{ do } c] &= \mathcal{C}[\text{if } b \text{ then } (c; \text{while } b \text{ do } c) \text{ else skip}] \\ &= \text{cond}(\mathcal{B}[b], \mathcal{C}[c; \text{while } b \text{ do } c], \mathcal{C}[\text{skip}]) \\ &= \text{cond}(\mathcal{B}[b], \mathcal{C}[\text{while } b \text{ do } c] \circ \mathcal{C}[c], \text{id}_{\Sigma})\end{aligned}$$

- Abbreviating $f := \mathcal{C}[\text{while } b \text{ do } c]$ this yields:

$$f = \text{cond}(\mathcal{B}[b], f \circ \mathcal{C}[c], \text{id}_{\Sigma})$$

- Hence f must be a **solution** of this recursive equation
- Or: f must be a **fixpoint** of the mapping

$$\Phi : (\Sigma \rightarrow \Sigma) \rightarrow (\Sigma \rightarrow \Sigma) : f \mapsto \text{cond}(\mathcal{B}[b], f \circ \mathcal{C}[c], \text{id}_{\Sigma})$$

(since the equation can be stated as $f = \Phi(f)$)

But: fixpoint property not sufficient to obtain a well-defined semantics

Existence: there does not need to exist any fixpoint. Examples:

- ❶ $\phi_1 : \mathbb{N} \rightarrow \mathbb{N} : n \mapsto n + 1$ has no fixpoint
- ❷ $\Phi_1 : (\Sigma \rightarrow \Sigma) \rightarrow (\Sigma \rightarrow \Sigma) : f \mapsto \begin{cases} g_1 & \text{if } f = g_2 \\ g_2 & \text{otherwise} \end{cases}$
(where $g_1 \neq g_2$) has no fixpoint

Solution: in our setting, **fixpoints always exist**

Uniqueness: there might exist several fixpoints. Examples:

- ❶ $\phi_2 : \mathbb{N} \rightarrow \mathbb{N} : n \mapsto n^3$ has fixpoints $\{0, 1\}$
- ❷ every state transformation f is a fixpoint of
 $\Phi_2 : (\Sigma \rightarrow \Sigma) \rightarrow (\Sigma \rightarrow \Sigma) : f \mapsto f$

Solution: guaranteed by **choosing a special fixpoint**

- 1 Repetition: Denotational Semantics
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- Let $b \in BExp$ and $c \in Cmd$
- Let $\Phi(f) := \text{cond}(\mathcal{B}[[b]], f \circ \mathcal{C}[[c]], \text{id}_\Sigma)$
- Let $f_0 : \Sigma \rightarrow \Sigma$ be a fixpoint of Φ , i.e., $\Phi(f_0) = f_0$
- Given some initial state $\sigma_0 \in \Sigma$, we will distinguish the following cases:
 - ① loop **while** b do c terminates after n iterations ($n \in \mathbb{N}$)
 - ② body c diverges in the n th iteration (since it contains a non-terminating **while** statement)
 - ③ loop **while** b do c itself diverges

Case 1: Termination of Loop

- Loop `while b do c` terminates after n iterations ($n \in \mathbb{N}$)
- Formally: there exist $\sigma_1, \dots, \sigma_n \in \Sigma$ such that

$$\begin{aligned} \mathfrak{B}[[b]]\sigma_i &= \begin{cases} \text{true} & \text{if } 0 \leq i < n \\ \text{false} & \text{if } i = n \end{cases} \quad \text{and} \\ \mathfrak{C}[[c]]\sigma_i &= \sigma_{i+1} \quad \text{for every } 0 \leq i < n \end{aligned}$$

- Now the definition

$\Phi(f) := \text{cond}(\mathfrak{B}[[b]], f \circ \mathfrak{C}[[c]], \text{id}_\Sigma)$ implies, for every $0 \leq i < n$,

$$\begin{aligned} \Phi(f_0)(\sigma_i) &= (f_0 \circ \mathfrak{C}[[c]])(\sigma_i) \quad \text{since } \mathfrak{B}[[b]]\sigma_i = \text{true} \\ &= f_0(\sigma_{i+1}) \quad \text{and} \end{aligned}$$

$$\Phi(f_0)(\sigma_n) = \sigma_n \quad \text{since } \mathfrak{B}[[b]]\sigma_n = \text{false}$$

- Since $\Phi(f_0) = f_0$ it follows that

$$f_0(\sigma_i) = \begin{cases} f_0(\sigma_{i+1}) & \text{if } 0 \leq i < n \\ \sigma_n & \text{if } i = n \end{cases}$$

and hence

$$f_0(\sigma_0) = f_0(\sigma_1) = \dots f_0(\sigma_n) = \sigma_n$$

\Rightarrow All fixpoints coincide on σ_0 !

Case 2: Divergence of Body

- Body c diverges in the n th iteration (since it contains a non-terminating **while** statement)
- Formally: there exist $\sigma_1, \dots, \sigma_{n-1} \in \Sigma$ such that

$$\begin{aligned} \mathfrak{B}[b]\sigma_i &= \text{true} && \text{for every } 0 \leq i < n \text{ and} \\ \mathfrak{C}[c]\sigma_i &= \begin{cases} \sigma_{i+1} & \text{if } 0 \leq i \leq n-2 \\ \text{undefined} & \text{if } i = n-1 \end{cases} \end{aligned}$$

- Just as in the previous case (setting $\sigma_n := \text{undefined}$) it follows that

$$f_0(\sigma_0) = \text{undefined}$$

\Rightarrow Again all fixpoints coincide on σ_0 !

Case 3: Divergence of Loop

- Loop `while b do c` diverges
- Formally: there exist $\sigma_1, \sigma_2, \dots \in \Sigma$ such that

$$\begin{aligned} \mathfrak{B}[[b]]\sigma_i &= \text{true} & \text{and} \\ \mathfrak{C}[[c]]\sigma_i &= \sigma_{i+1} & \text{for every } i \in \mathbb{N} \end{aligned}$$

- Here only derivable:

$$f_0(\sigma_0) = f_0(\sigma_i) \quad \text{for every } i \in \mathbb{N}$$

\Rightarrow Value of $f_0(\sigma_0)$ not determined!

Summary

For $\Phi(f_0) = f_0$ and initial state $\sigma_0 \in \Sigma$, case distinction yields:

- ① Loop **while** b **do** c terminates after n iterations ($n \in \mathbb{N}$)
 $\implies f_0(\sigma_0) = \sigma_n$
- ② Body c diverges in the n th iteration
 $\implies f_0(\sigma_0) = \text{undefined}$
- ③ Loop **while** b **do** c diverges
 \implies no condition on f_0 (only $f_0(\sigma_0) = f_0(\sigma_i)$ for every $i \in \mathbb{N}$)
- Not surprising since, e.g., **while true do skip** yields for every $f : \Sigma \rightarrow \Sigma$

$$\Phi(f) = \text{cond}(\mathcal{B}[\text{true}], f \circ \mathcal{C}[\text{skip}], \text{id}_\Sigma) = f$$

- On the other hand, our operational understanding requires, for every $\sigma_0 \in \Sigma$,

$$\mathcal{C}[\text{while true do skip}]\sigma_0 = \text{undefined}$$

Conclusion

$\text{fix}(\Phi)$ is the least defined fixpoint of Φ .

To use fixpoint theory, the notion of “least defined” has to be made precise.

- Given $f, g : \Sigma \rightarrow \Sigma$, let

$$f \sqsubseteq g \iff \text{for every } \sigma, \sigma' \in \Sigma : f(\sigma) = \sigma' \implies g(\sigma) = \sigma'$$

(g is “at least as defined” as f)

- Equivalent to requiring

$$\text{graph}(f) \subseteq \text{graph}(g)$$

where

$$\text{graph}(h) := \{(\sigma, \sigma') \mid \sigma \in \Sigma, \sigma' = h(\sigma) \text{ defined}\} \subseteq \Sigma \times \Sigma$$

for every $h : \Sigma \rightarrow \Sigma$

Example 5.1

Let $x \in Var$ be fixed, and let $f_0, f_1, f_2, f_3 : \Sigma \rightarrow \Sigma$ be given by

$$\begin{aligned} f_0(\sigma) &:= \text{undefined} \\ f_1(\sigma) &:= \begin{cases} \sigma & \text{if } \sigma(x) \text{ even} \\ \text{undefined} & \text{otherwise} \end{cases} \\ f_2(\sigma) &:= \begin{cases} \sigma & \text{if } \sigma(x) \text{ odd} \\ \text{undefined} & \text{otherwise} \end{cases} \\ f_3(\sigma) &:= \sigma \end{aligned}$$

This implies $f_0 \sqsubseteq f_1 \sqsubseteq f_3$, $f_0 \sqsubseteq f_2 \sqsubseteq f_3$, $f_1 \not\sqsubseteq f_2$, and $f_2 \not\sqsubseteq f_1$

Characterization of $\text{fix}(\Phi)$ II

Now $\text{fix}(\Phi)$ can be characterized by:

- $\text{fix}(\Phi)$ is a **fixpoint** of Φ , i.e.,

$$\Phi(\text{fix}(\Phi)) = \text{fix}(\Phi)$$

- $\text{fix}(\Phi)$ is **minimal** with respect to \sqsubseteq , i.e., for every $f_0 : \Sigma \rightarrow \Sigma$ such that $\Phi(f_0) = f_0$,

$$\text{fix}(\Phi) \sqsubseteq f_0$$

Example 5.2

For `while true do skip` we obtain for every $f : \Sigma \rightarrow \Sigma$:

$$\Phi(f) = \text{cond}(\mathcal{B}[\text{true}], f \circ \mathcal{C}[\text{skip}], \text{id}_\Sigma) = f$$

$\implies \text{fix}(\Phi) = f_\emptyset$ where $f_\emptyset(\sigma) := \text{undefined}$ for every $\sigma \in \Sigma$
(that is, $\text{graph}(f_\emptyset) = \emptyset$)

Goals:

- Prove **existence** of $\text{fix}(\Phi)$ for $\Phi(f) = \text{cond}(\mathfrak{B}[[b]], f \circ \mathfrak{C}[[c]], \text{id}_\Sigma)$
- Show how it can be **“computed”** (more exactly: approximated)

Sufficient conditions:

on domain $\Sigma \rightarrow \Sigma$: **chain-complete partial order**

on function Φ : **continuity**

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Definition 5.3 (Partial order)

A **partial order (PO)** (D, \sqsubseteq) consists of a set D , called **domain**, and of a relation $\sqsubseteq \subseteq D \times D$ such that, for every $d_1, d_2, d_3 \in D$,

reflexivity: $d_1 \sqsubseteq d_1$

transitivity: $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_3 \implies d_1 \sqsubseteq d_3$

antisymmetry: $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_1 \implies d_1 = d_2$

It is called **total** if, in addition, always $d_1 \sqsubseteq d_2$ or $d_2 \sqsubseteq d_1$.

Example 5.4

- ❶ (\mathbb{N}, \leq) is a total partial order
- ❷ $(2^{\mathbb{N}}, \subseteq)$ is a (non-total) partial order
- ❸ $(\mathbb{N}, <)$ is not a partial order (since not reflexive)

Application to fix(Φ) I

Lemma 5.5

$(\Sigma \rightarrow \Sigma, \sqsubseteq)$ is a partial order.

Proof.

see exercise □

Chains and Least Upper Bounds

Definition 5.6 (Chain, (least) upper bound)

Let (D, \sqsubseteq) be a partial order and $S \subseteq D$.

- 1 S is called a **chain** in D if, for every $s_1, s_2 \in S$,
$$s_1 \sqsubseteq s_2 \text{ or } s_2 \sqsubseteq s_1$$

(that is, S is a totally ordered subset of D).
- 2 An element $d \in D$ is called an **upper bound** of S if $s \sqsubseteq d$ for every $s \in S$ (notation: $S \sqsubseteq d$).
- 3 An upper bound d of S is called **least upper bound (LUB)** or **supremum** of S if $d \sqsubseteq d'$ for every upper bound d' of S (notation: $d = \bigsqcup S$).

Example 5.7

- 1 Every subset $S \subseteq \mathbb{N}$ is a chain in (\mathbb{N}, \leq) .
It has a LUB (its greatest element) iff it is finite.
- 2 $\{\emptyset, \{0\}, \{0, 1\}, \dots\}$ is a chain in $(2^{\mathbb{N}}, \subseteq)$ with LUB \mathbb{N} .

Definition 5.8 (Chain completeness)

A partial order is called **chain complete (CCPO)** if every of its chains has a least upper bound.

Example 5.9

- ❶ $(2^{\mathbb{N}}, \subseteq)$ is a CCPO with $\bigsqcup S = \bigcup_{M \in S} M$ for every chain $S \subseteq 2^{\mathbb{N}}$.
- ❷ (\mathbb{N}, \leq) is not chain complete
(since, e.g., the chain \mathbb{N} has no upper bound).

Corollary 5.10

Every CCPO has a least element $\bigsqcup \emptyset$.

Proof.

Let (D, \sqsubseteq) be a CCPO.

- By definition, \emptyset is a chain in D .
- By definition, every $d \in D$ is an upper bound of \emptyset .
- Thus $\bigsqcup \emptyset$ exists and is the least element of D .



Lemma 5.11

- $(\Sigma \multimap \Sigma, \sqsubseteq)$ is a CCPO with least element f_\emptyset where $\text{graph}(f_\emptyset) = \emptyset$.
- In particular, for every chain $S \subseteq \Sigma \multimap \Sigma$,

$$\text{graph} \left(\bigsqcup S \right) = \bigcup_{f \in S} \text{graph}(f).$$

Proof.

on the board

