Semantics and Verification of Software

Lecture 17: Dataflow Analysis IV (Equation Solving)

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Outline

Complete Lattices

Definition (Complete lattice)

A complete lattice is a partial order $(D, \sqsubseteq)$ such that all subsets of D have least upper as well as greatest lower bounds. In this case,

$$\begin{aligned} \bot &:= \bigsqcup \emptyset = \bigsqcap D \text{ and} \\ \top &:= \bigsqcap \emptyset = \bigsqcup D \end{aligned}$$

denote the least and the greatest element of D, respectively.

Example

1. (Available Expressions) $(D, \sqsubseteq) = (2^{AExp_c}, \supseteq)$ is a complete lattice with $\bot = AExp_c$ and $\top = \emptyset$
2. (Live Variables) $(D, \sqsubseteq) = (2^{Var_c}, \subseteq)$ is a complete lattice with $\bot = \emptyset$ and $\top = Var_c$

Chains

Chains represent the approximation of the analysis information.

Definition (Chain; repetition of Def. 6.4 and 6.6)

Let $(D, \sqsubseteq)$ be a partial order.

1. A subset $S \subseteq D$ is called a chain in D if, for every $s_1, s_2 \in S$,
$$s_1 \sqsubseteq s_2 \text{ or } s_2 \sqsubseteq s_1$$
(that is, S is a totally ordered subset of D).
2. $(D, \sqsubseteq)$ is called chain complete (CCPO) if each of its chains has a least upper bound.
3. $(D, \sqsubseteq)$ satisfies the Ascending Chain Condition (ACC) if each ascending chain $d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ eventually stabilizes, i.e., there exists $n \in \mathbb{N}$ such that $d_n = d_{n+1} = \ldots$

Corollary

Complete lattices are CCPOs.

Monotonicity of Functions

Transfer functions formalize the impact of a block in the program on the analysis information.

Definition (Monotonicity; repetition of Def. 7.1)

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be partial orders, and let $F : D \to D'$. F is called monotonic (w.r.t. $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$) if, for every $d_1, d_2 \in D$,

$$d_1 \sqsubseteq d_2 \implies F(d_1) \sqsubseteq' F(d_2).$$

Example

1. (Available Expressions) $(D, \sqsubseteq) = (2^{AExp_c}, \supseteq)$
   Each transfer function $\varphi_{l'}(A) := (A \setminus \mathsf{kill}_{\mathsf{AE}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{AE}}(B^{l'})$ is monotonic
2. (Live Variables) $(D, \sqsubseteq) = (2^{Var_c}, \subseteq)$
   Each transfer function $\varphi_{l'}(V) := (V \setminus \mathsf{kill}_{\mathsf{LV}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{LV}}(B^{l'})$ is monotonic

Fixpoints

Theorem (Fixpoint Theorem; repetition of Thm. 7.7)

Let $(D, \sqsubseteq)$ be a CCPO and $F : D \to D$ continuous. Then

$$\mathsf{fix}(F) := \bigsqcup \{F^n (\bigsqcup \emptyset) \mid n \in \mathbb{N}\}$$

is the least fixpoint of F.

Definition (Continuity; repetition of Def. 7.5)

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be CCPOs and $F : D \to D'$ monotonic. Then F is called continuous (w.r.t. $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$) if, for every non-empty chain $S \subseteq D$,

$$F (\bigsqcup S) = \bigsqcup F(S).$$

Corollary

Monotonic functions on partial orders that satisfy ACC are continuous.

Dataflow Systems I

Definition (Dataflow system)

A dataflow system $S = (L, E, F, (D, \sqsubseteq), \iota, \varphi)$ consists of

- a finite set of (program) labels L (here: L_c),
- a set of extremal labels $E \subseteq L$ (here: $\{\mathsf{init}(c)\}$ or $\mathsf{final}(c)$),
- a flow relation $F \subseteq L \times L$ (here: $\mathsf{flow}(c)$ or $\mathsf{flow}^R(c)$),
- a complete lattice $(D, \sqsubseteq)$ that satisfies ACC (with LUB operator $\bigsqcup$ and least element $\bot$),
- an extremal value $\iota \in D$ (for the extremal labels), and
- a collection of monotonic transfer functions $\{\varphi_l \mid l \in L\}$ of type $\varphi_l : D \to D$.

Dataflow Systems II

Example

Problem	**Available Expressions**	**Live Variables**
E	$\{\mathsf{init}(c)\}$	$\mathsf{final}(c)$
F	$\mathsf{flow}(c)$	$\mathsf{flow}^R(c)$
D	2^{AExp_c}	2^{Var_c}
$\sqsubseteq$	$\supseteq$	$\subseteq$
$\bigsqcup$	$\bigcap$	$\bigcup$
$\perp$	$AExp_c$	$\emptyset$
ι	$\emptyset$	Var_c
φ_l	$\varphi_l(d) = (d \setminus \mathsf{kill}(B^l)) \cup \mathsf{gen}(B^l)$	

Outline

The Equation System

Definition 17.1 (Dataflow equation system)

Let $S = (L, E, F, (D, \sqsubseteq), \iota, \varphi)$ be a dataflow system. S defines the following equation system over the set of variables $\{\mathsf{AI}_l \mid l \in L\}$:

$$\mathsf{AI}_l = \begin{cases} \iota & \text{if } l \in E \\ \bigsqcup\{\varphi_{l'}(\mathsf{AI}_{l'}) \mid (l', l) \in F\} & \text{otherwise} \end{cases}$$

The Functional

Just as in the denotational semantics of `while` loops, the equation system determines a functional whose fixpoints are exactly the solutions of the equation system.

Definition 17.2 (Dataflow functional)

The equation system of a dataflow system $S = (L, E, F, (D, \sqsubseteq), \iota, \varphi)$ induces a functional

$$\Phi_S : D^n \to D^n : (d_{l_1}, \dots, d_{l_n}) \mapsto (d'_{l_1}, \dots, d'_{l_n})$$

where $L = \{l_1, \dots, l_n\}$ and, for each $1 \leq i \leq n$,

$$d'_{l_i} := \begin{cases} \iota & \text{if } l_i \in E \\ \bigsqcup\{\varphi_{l'}(d_{l'}) \mid (l', l_i) \in F\} & \text{otherwise} \end{cases}$$

Fixpoint Iteration I

Remarks:

- $(D, \sqsubseteq)$ being a complete lattice ensures that Φ_S is well defined

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- $(D, \sqsubseteq)$ being a complete lattice ensures that Φ_S is well defined
- $(d_1, \ldots, d_n)$ is a solution of the equation system iff it is a fixpoint of Φ_S
- If $(D, \sqsubseteq)$ is a complete lattice satisfying ACC, then so is $(D^n, \sqsubseteq^n)$ (where $(d_1, \ldots, d_n) \sqsubseteq^n (d'_1, \ldots, d'_n)$ iff $d_i \sqsubseteq d'_i$ for every $1 \leq i \leq n$)

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- Every transfer function φ_l monotonic in D
  $\Longrightarrow$ Φ_S monotonic in D^n

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- Every transfer function φ_l monotonic in D
  $\implies$ Φ_S monotonic in D^n
- Thus the (least) fixpoint is effectively computable by iteration:

$$\mathsf{fix}(\Phi_S) = \bigsqcup \{\Phi_S^i(\bot_{D^n}) \mid i \in \mathbb{N}\}$$

where $\bot_{D^n} = \underbrace{(\bot_D, \ldots, \bot_D)}_{n \text{ times}}$

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- $(D, \sqsubseteq)$ being a complete lattice ensures that Φ_S is well defined
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  where $\bot_{D^n} = (\underbrace{\bot_D, \ldots, \bot_D}_{n \text{ times}})$
- If maximal length of chains in D is m
  $\implies$ maximal length of chains in D^n is $m \cdot n$
  $\implies$ fixpoint iteration requires at most $m \cdot n$ steps

Fixpoint Iteration II

Example 17.3 (Available Expressions; cf. Example 14.9)

Program:

```
c = [x := a+b]^1;
    [y := a*b]^2;
    while [y > a+b]^3 do
      [a := a+1]^4;
      [x := a+b]^5
```

Fixpoint Iteration II

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$$\begin{array}{l} c = [\texttt{x := a+b}]^1; \\ \quad\; [\texttt{y := a*b}]^2; \\ \quad\; \texttt{while } [\texttt{y > a+b}]^3 \texttt{ do} \\ \quad\quad [\texttt{a := a+1}]^4; \\ \quad\quad [\texttt{x := a+b}]^5 \end{array}$$

Equation system:

$$\begin{aligned} \mathsf{AE}_1 &= \emptyset \\ \mathsf{AE}_2 &= \mathsf{AE}_1 \cup \{\texttt{a+b}\} \\ \mathsf{AE}_3 &= (\mathsf{AE}_2 \cup \{\texttt{a*b}\}) \cap (\mathsf{AE}_5 \cup \{\texttt{a+b}\}) \\ \mathsf{AE}_4 &= \mathsf{AE}_3 \cup \{\texttt{a+b}\} \\ \mathsf{AE}_5 &= \mathsf{AE}_4 \setminus \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\} \end{aligned}$$

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\end{aligned}$$

Fixpoint iteration:

i	1	2	3	4	5
0	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$	$AExp_c$

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1	$\emptyset$	$AExp_c$	$AExp_c$	$AExp_c$	$\emptyset$

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1	$\emptyset$	$AExp_c$	$AExp_c$	$AExp_c$	$\emptyset$
2	$\emptyset$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$AExp_c$	$\emptyset$

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1	$\emptyset$	$AExp_c$	$AExp_c$	$AExp_c$	$\emptyset$
2	$\emptyset$	{a+b}	{a+b}	$AExp_c$	$\emptyset$
3	$\emptyset$	{a+b}	{a+b}	{a+b}	$\emptyset$

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2	$\emptyset$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$AExp_c$	$\emptyset$
3	$\emptyset$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$\emptyset$
4	$\emptyset$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$\{\texttt{a+b}\}$	$\emptyset$

Fixpoint Iteration III

Example 17.4 (Live Variables; cf. Example 15.3)

Program:

$[\texttt{x := 2}]^1$;$[\texttt{y := 4}]^2$;
$[\texttt{x := 1}]^3$;
if $[\texttt{y > 0}]^4$ then
  $[\texttt{z := x}]^5$
else
  $[\texttt{z := y*y}]^6$;
$[\texttt{x := z}]^7$

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Equation system:

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\begin{aligned}
\mathsf{LV}_1 &= \mathsf{LV}_2 \setminus \{y\}\\
\mathsf{LV}_2 &= \mathsf{LV}_3 \setminus \{x\}\\
\mathsf{LV}_3 &= \mathsf{LV}_4 \cup \{y\}\\
\mathsf{LV}_4 &= ((\mathsf{LV}_5 \setminus \{z\}) \cup \{x\}) \cup ((\mathsf{LV}_6 \setminus \{z\}) \cup \{y\})\\
\mathsf{LV}_5 &= (\mathsf{LV}_7 \setminus \{x\}) \cup \{z\}\\
\mathsf{LV}_6 &= (\mathsf{LV}_7 \setminus \{x\}) \cup \{z\}\\
\mathsf{LV}_7 &= \{x, y, z\}
\end{aligned}
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1	$\emptyset$	$\emptyset$	$\{\mathtt{y}\}$	$\{\mathtt{x}, \mathtt{y}\}$	$\{\mathtt{z}\}$	$\{\mathtt{z}\}$	$\{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$

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1	$\emptyset$	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{z\}$	$\{z\}$	$\{x, y, z\}$
2	$\emptyset$	$\{y\}$	$\{x, y\}$	$\{x, y\}$	$\{y, z\}$	$\{y, z\}$	$\{x, y, z\}$

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Uniqueness of Solutions

Just as in the denotational semantics of `while` loops, solutions of dataflow equation systems are not unique.

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1. Available Expressions: consider

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[z := x+y]^1;
while [true]^2 do
  [skip]^3;
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[z := x+y]^1;
while [true]^2 do
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$$\Longrightarrow \begin{aligned} \mathsf{AE}_1 &= \emptyset \\ \mathsf{AE}_2 &= (\mathsf{AE}_1 \cup \{\texttt{x+y}\}) \cap \mathsf{AE}_3 \\ \mathsf{AE}_3 &= \mathsf{AE}_2 \end{aligned}$$

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[z := x+y]¹;
while [true]² do
   [skip]³;
```

$$\Longrightarrow \begin{aligned} \mathsf{AE}_1 &= \emptyset \\ \mathsf{AE}_2 &= (\mathsf{AE}_1 \cup \{\texttt{x+y}\}) \cap \mathsf{AE}_3 \\ \mathsf{AE}_3 &= \mathsf{AE}_2 \end{aligned}$$

$$\Longrightarrow \begin{aligned} \mathsf{AE}_1 &= \emptyset \\ \mathsf{AE}_2 &= \{\texttt{x+y}\} \cap \mathsf{AE}_3 \\ \mathsf{AE}_3 &= \mathsf{AE}_2 \end{aligned}$$

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  [skip]^3;
```

$$\Longrightarrow \begin{aligned} \mathsf{AE}_1 &= \emptyset \\ \mathsf{AE}_2 &= (\mathsf{AE}_1 \cup \{\texttt{x+y}\}) \cap \mathsf{AE}_3 \\ \mathsf{AE}_3 &= \mathsf{AE}_2 \end{aligned}$$

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$\Longrightarrow$ Solutions: $\mathsf{AE}_1 = \mathsf{AE}_2 = \mathsf{AE}_3 = \emptyset$ or
$\mathsf{AE}_1 = \emptyset, \mathsf{AE}_2 = \mathsf{AE}_3 = \{\texttt{x+y}\}$

Uniqueness of Solutions

Just as in the denotational semantics of `while` loops, solutions of dataflow equation systems are not unique.

Example 17.5

1. Available Expressions: consider

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[z := x+y]^1;
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Here: greatest solution $\{\mathtt{x+y}\}$ (maximal potential for optimization)

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2. Live Variables: see Exercise 9.3