Semantics and Verification of Software

Lecture 12: Axiomatic Semantics of WHILE III (Correctness of Hoare Logic)

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Summer Semester 2010

Outline

Partial Correctness Properties

Definition (Partial correctness properties)

Let $A, B \in Assn$ and $c \in Cmd$.

- An expression of the form $\{A\}\, c\, \{B\}$ is called a partial correctness property with precondition A and postcondition B.
- Given $\sigma \in \Sigma_\perp$ and $I \in Int$, we let

$$\sigma \models^I \{A\}\, c\, \{B\}$$

  if $\sigma \models^I A$ implies $\mathfrak{C}[\![c]\!]\sigma \models^I B$
  (or equivalently: $\sigma \in A^I \implies \mathfrak{C}[\![c]\!]\sigma \in B^I$).
- $\{A\}\, c\, \{B\}$ is called valid in I (notation: $\models^I \{A\}\, c\, \{B\}$) if $\sigma \models^I \{A\}\, c\, \{B\}$ for every $\sigma \in \Sigma_\perp$ (or equivalently: $\mathfrak{C}[\![c]\!]A^I \subseteq B^I$).
- $\{A\}\, c\, \{B\}$ is called valid (notation: $\models \{A\}\, c\, \{B\}$) if $\models^I \{A\}\, c\, \{B\}$ for every $I \in Int$.

Hoare Logic

Goal: syntactic derivation of valid partial correctness properties

Definition (Hoare Logic)

The Hoare rules are given by

$$(\text{skip})\ \frac{}{\{A\}\,\texttt{skip}\,\{A\}} \qquad (\text{asgn})\ \frac{}{\{A[x \mapsto a]\}\,x\,\texttt{:=}a\,\{A\}}$$

$$(\text{seq})\ \frac{\{A\}\,c_1\,\{C\} \quad \{C\}\,c_2\,\{B\}}{\{A\}\,c_1\,\texttt{;}c_2\,\{B\}} \qquad (\text{if})\ \frac{\{A \wedge b\}\,c_1\,\{B\} \quad \{A \wedge \neg b\}\,c_2\,\{B\}}{\{A\}\,\texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\,\{B\}}$$

$$(\text{while})\ \frac{\{A \wedge b\}\,c\,\{A\}}{\{A\}\,\texttt{while}\ b\ \texttt{do}\ c\,\{A \wedge \neg b\}}$$

$$(\text{cons})\ \frac{\models (A \implies A') \quad \{A'\}\,c\,\{B'\} \quad \models (B' \implies B)}{\{A\}\,c\,\{B\}}$$

A partial correctness property is provable (notation: $\vdash \{A\}\,c\,\{B\}$) if it is derivable by the Hoare rules. In case of (while), A is called a (loop) invariant.

Here $A[x \mapsto a]$ denotes the syntactic replacement of every occurrence of x by a in A.

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Soundness of Hoare Logic I

Soundness: no wrong propositions can be derived, i.e., every (syntactically) provable partial correctness property is also (semantically) valid

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For the corresponding proof we use:

Lemma 12.1 (Substitution lemma)

For every $A \in Assn$, $x \in Var$, $a \in AExp$, $\sigma \in \Sigma$, and $I \in Int$:

$$\sigma \models^I A[x \mapsto a] \iff \sigma[x \mapsto \mathfrak{A}[\![a]\!]\sigma] \models^I A.$$

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Proof.

by induction over $A \in Assn$ (omitted) □

Theorem 12.2 (Soundness of Hoare Logic)

For every partial correctness property $\{A\}\,c\,\{B\}$,

$$\vdash \{A\}\,c\,\{B\} \quad\Longrightarrow\quad \models \{A\}\,c\,\{B\}.$$

Soundness of Hoare Logic II

Theorem 12.2 (Soundness of Hoare Logic)

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$$\vdash \{A\}\,c\,\{B\} \quad\Longrightarrow\quad \models \{A\}\,c\,\{B\}.$$

Proof.

Let $\vdash \{A\}\,c\,\{B\}$. By induction over the structure of the corresponding proof tree we show that, for every $\sigma \in \Sigma$ and $I \in Int$ such that $\sigma \models^I A$, $\mathfrak{C}[\![c]\!]\sigma \models^I B$ (on the board).
(If $\sigma = \bot$, then $\mathfrak{C}[\![c]\!]\sigma = \bot \models^I B$ holds trivially.) □

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Incompleteness of Hoare Logic I

Soundness: only valid partial correctness properties are provable ✓

Completeness: all valid partial correctness properties are systematically derivable ↯

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Theorem 12.3 (Gödel's Incompleteness Theorem)

The set of all valid assertions

$$\{A \in Assn \mid \models A\}$$

is not recursively enumerable, i.e., there exists no proof system for $Assn$ *in which all valid assertions are systematically derivable.*

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Proof.

see [Winskel 1996, p. 110 ff] □

Incompleteness of Hoare Logic II

Corollary 12.4

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Proof.

Given $A \in Assn$, $\models A$ is obviously equivalent to $\{\texttt{true}\}\,\texttt{skip}\,\{A\}$. Thus the enumerability of all valid partial correctness properties would imply the enumerability of all valid assertions. □

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Remark: alternative proof (using computability theory):
$\{\mathsf{true}\}\,c\,\{\mathsf{false}\}$ is valid iff c does not terminate on any input state. But the set of all non-terminating WHILE statements is not enumerable.