Semantics and Verification of Software

Lecture 18: Dataflow Analysis I (Introduction & Available Expressions Analysis)

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Outline

1. Preliminaries on Dataflow Analysis

2. An Example: Available Expressions Analysis

Dataflow Analysis: the Approach

- Traditional form of program analysis

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- Idea: describe how analysis information flows through program

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- Traditional form of program analysis
- Idea: describe how analysis information flows through program
- Distinctions:

  direction of flow: forward vs. backward analyses
  procedures: interprocedural vs. intraprocedural analyses
  quantification over paths: may (union) vs. must (intersection) analyses
  dependence on statement order: flow-sensitive vs. flow-insensitive analyses
  distinction of procedure calls: context-sensitive vs. context-insensitive analyses

Labelled Programs

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  - assignments
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Definition 18.1 (Labelled WHILE programs)

The syntax of labelled WHILE programs is defined by the following context-free grammar:

$$\begin{aligned}
a &::= z \mid x \mid a_1\texttt{+}a_2 \mid a_1\texttt{-}a_2 \mid a_1\texttt{*}a_2 \in AExp \\
b &::= t \mid a_1\texttt{=}a_2 \mid a_1\texttt{>}a_2 \mid \neg b \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \in BExp \\
c &::= [\texttt{skip}]^l \mid [x \texttt{ := } a]^l \mid c_1\texttt{;}c_2 \mid \\
&\qquad \texttt{if } [b]^l \texttt{ then } c_1 \texttt{ else } c_2 \mid \texttt{while } [b]^l \texttt{ do } c \in Cmd
\end{aligned}$$

Here all labels in a statement $c \in Cmd$ are assumed to be distinct.

A WHILE Program

Example 18.2

```
x := 6;
y := 7;
z := 0;
while x > 0 do
  x := x - 1;
  v := y;
  while v > 0 do
    v := v - 1;
    z := z + 1;
```

A WHILE Program with Labels

Example 18.2

$$
\begin{array}{l}
[\texttt{x := 6}]^1; \\
[\texttt{y := 7}]^2; \\
[\texttt{z := 0}]^3; \\
\texttt{while } [\texttt{x > 0}]^4 \texttt{ do} \\
\quad [\texttt{x := x - 1}]^5; \\
\quad [\texttt{v := y}]^6; \\
\quad \texttt{while } [\texttt{v > 0}]^7 \texttt{ do} \\
\qquad [\texttt{v := v - 1}]^8; \\
\qquad [\texttt{z := z + 1}]^9
\end{array}
$$

Representing Control Flow I

Every (labelled) statement has a single entry (given by the initial label) and generally multiple exits (given by the final labels):

Definition 18.3 (Initial and final labels)

The mapping $\mathsf{init} : Cmd \to L$ returns the initial label of a statement:

$$\begin{aligned}
\mathsf{init}([\texttt{skip}]^l) &:= l \\
\mathsf{init}([x \texttt{ := } a]^l) &:= l \\
\mathsf{init}(c_1 \texttt{;} c_2) &:= \mathsf{init}(c_1) \\
\mathsf{init}(\texttt{if } [b]^l \texttt{ then } c_1 \texttt{ else } c_2) &:= l \\
\mathsf{init}(\texttt{while } [b]^l \texttt{ do } c) &:= l
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\mathsf{init}(\mathtt{if}\ [b]^l\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2) &:= l \\
\mathsf{init}(\mathtt{while}\ [b]^l\ \mathtt{do}\ c) &:= l
\end{aligned}$$

The mapping $\mathsf{final} : \mathit{Cmd} \to 2^L$ returns the set of final labels of a statement:

$$\begin{aligned}
\mathsf{final}([\mathtt{skip}]^l) &:= \{l\} \\
\mathsf{final}([x \texttt{ := } a]^l) &:= \{l\} \\
\mathsf{final}(c_1 \,;\, c_2) &:= \mathsf{final}(c_2) \\
\mathsf{final}(\mathtt{if}\ [b]^l\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2) &:= \mathsf{final}(c_1) \cup \mathsf{final}(c_2) \\
\mathsf{final}(\mathtt{while}\ [b]^l\ \mathtt{do}\ c) &:= \{l\}
\end{aligned}$$

Representing Control Flow II

Definition 18.4 (Flow relation)

Given a statement $c \in Cmd$, the (control) flow relation $\mathsf{flow}(c) \subseteq L \times L$ is defined by

$$
\begin{aligned}
\mathsf{flow}([\mathtt{skip}]^l) &:= \emptyset \\
\mathsf{flow}([x \;\mathtt{:=}\; a]^l) &:= \emptyset \\
\mathsf{flow}(c_1\,\mathtt{;}\,c_2) &:= \mathsf{flow}(c_1) \cup \mathsf{flow}(c_2) \cup \\
&\ \{(l, \mathsf{init}(c_2)) \mid l \in \mathsf{final}(c_1)\} \\
\mathsf{flow}(\mathtt{if}\ [b]^l\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2) &:= \mathsf{flow}(c_1) \cup \mathsf{flow}(c_2) \cup \\
&\ \{(l, \mathsf{init}(c_1)), (l, \mathsf{init}(c_2))\} \\
\mathsf{flow}(\mathtt{while}\ [b]^l\ \mathtt{do}\ c) &:= \mathsf{flow}(c) \cup \{(l, \mathsf{init}(c))\} \cup \\
&\ \{(l', l) \mid l' \in \mathsf{final}(c)\}
\end{aligned}
$$

Representing Control Flow III

Example 18.5

$c = [\texttt{z := 1}]^1;$
$\quad \texttt{while}\ [\texttt{x > 0}]^2\ \texttt{do}$
$\quad\quad [\texttt{z := z*y}]^3;$
$\quad\quad [\texttt{x := x-1}]^4$

Representing Control Flow III

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\begin{aligned}
c = {} & [\texttt{z := 1}]^1; \\
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$$

$$
\begin{aligned}
\mathsf{init}(c) &= 1 \\
\mathsf{final}(c) &= \{2\} \\
\mathsf{flow}(c) &= \{(1,2),(2,3),(3,4),(4,2)\}
\end{aligned}
$$

Representing Control Flow III

Example 18.5

Visualization by (control) flow graph:

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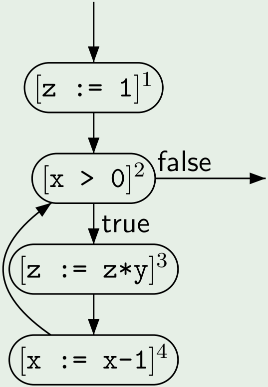

Representing Control Flow IV

- To simplify the presentation we will often assume that the program $c \in Cmd$ under consideration has an isolated entry, meaning that

$$\{l \in L \mid (l, \mathsf{init}(c)) \in \mathsf{flow}(c)\} = \emptyset$$

(which is the case when c does not start with a `while` loop)

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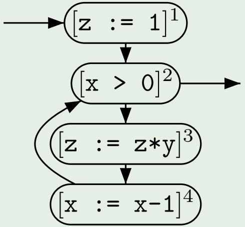
Example 18.6



has an isolated entry but not isolated exits

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Available Expressions Analysis

The goal of Available Expressions Analysis is to determine, for each program point, which (complex) expressions *must* have been computed, and not later modified, on all paths to the program point.

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Example 18.7 (Available Expressions Analysis)

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[x := a+b]^1;
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- `a+b` available at label 3
- `a+b` not available at label 5
- possible optimization:
  `while` $[\texttt{y > x}]^3$ `do`

Formalizing Available Expressions Analysis I

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$$\begin{aligned} \mathsf{kill}_{\mathsf{AE}}([\mathtt{skip}]^l) &:= \emptyset \\ \mathsf{kill}_{\mathsf{AE}}([x \ \mathtt{:=}\ a]^l) &:= \{a' \in AExp_c \mid x \in FV(a')\} \\ \mathsf{kill}_{\mathsf{AE}}([b]^l) &:= \emptyset \end{aligned}$$

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Example 18.8 ($\mathsf{kill}_{\mathsf{AE}}/\mathsf{gen}_{\mathsf{AE}}$ functions)

$c = [\texttt{x := a+b}]^1;$
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Formalizing Available Expressions Analysis II

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  where $\varphi_{l'} : 2^{AExp_c} \to 2^{AExp_c}$ denotes the transfer function of block $B^{l'}$, given by

$$\varphi_{l'}(A) := (A \setminus \mathsf{kill}_{\mathsf{AE}}(B^{l'})) \cup \mathsf{gen}_{\mathsf{AE}}(B^{l'})$$

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  must: $\bigcap$ in equation for AE_l

  flow-sensitive: results depending on order of assignments

- Later: solution not necessarily unique

  $\Longrightarrow$ choose greatest one

The Equation System II

Reminder:

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Equations:

$$\begin{aligned}
\mathsf{AE}_1 &= \emptyset \\
\mathsf{AE}_2 &= \varphi_1(\mathsf{AE}_1) = \mathsf{AE}_1 \cup \{\texttt{a+b}\} \\
\mathsf{AE}_3 &= \varphi_2(\mathsf{AE}_2) \cap \varphi_5(\mathsf{AE}_5) \\
&= (\mathsf{AE}_2 \cup \{\texttt{a*b}\}) \cap (\mathsf{AE}_5 \cup \{\texttt{a+b}\}) \\
\mathsf{AE}_4 &= \varphi_3(\mathsf{AE}_3) = \mathsf{AE}_3 \cup \{\texttt{a+b}\} \\
\mathsf{AE}_5 &= \varphi_4(\mathsf{AE}_4) = \mathsf{AE}_4 \setminus \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\}
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5	$\emptyset$	$\{\mathtt{a+b}\}$

Equations:
$$\begin{aligned} \mathsf{AE}_1 &= \emptyset \\ \mathsf{AE}_2 &= \varphi_1(\mathsf{AE}_1) = \mathsf{AE}_1 \cup \{\mathtt{a+b}\} \\ \mathsf{AE}_3 &= \varphi_2(\mathsf{AE}_2) \cap \varphi_5(\mathsf{AE}_5) \\ &= (\mathsf{AE}_2 \cup \{\mathtt{a*b}\}) \cap (\mathsf{AE}_5 \cup \{\mathtt{a+b}\}) \\ \mathsf{AE}_4 &= \varphi_3(\mathsf{AE}_3) = \mathsf{AE}_3 \cup \{\mathtt{a+b}\} \\ \mathsf{AE}_5 &= \varphi_4(\mathsf{AE}_4) = \mathsf{AE}_4 \setminus \{\mathtt{a+b}, \mathtt{a*b}, \mathtt{a+1}\} \end{aligned}$$

Solution:
$$\begin{aligned} \mathsf{AE}_1 &= \emptyset \\ \mathsf{AE}_2 &= \{\mathtt{a+b}\} \\ \mathsf{AE}_3 &= \{\mathtt{a+b}\} \\ \mathsf{AE}_4 &= \{\mathtt{a+b}\} \\ \mathsf{AE}_5 &= \emptyset \end{aligned}$$