Semantics and Verification of Software

Lecture 11: Axiomatic Semantics of WHILE III (Correctness of Hoare Logic)

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Outline

Partial Correctness Properties

Definition (Partial correctness properties)

Let $A, B \in Assn$ and $c \in Cmd$.

- An expression of the form $\{A\}\, c\, \{B\}$ is called a partial correctness property with precondition A and postcondition B.
- Given $\sigma \in \Sigma_\perp$ and $I \in Int$, we let

  $$\sigma \models^I \{A\}\, c\, \{B\}$$

  if $\sigma \models^I A$ implies $\mathfrak{C}[\![c]\!]\sigma \models^I B$
  (or equivalently: $\sigma \in A^I \implies \mathfrak{C}[\![c]\!]\sigma \in B^I$).
- $\{A\}\, c\, \{B\}$ is called valid in I (notation: $\models^I \{A\}\, c\, \{B\}$) if $\sigma \models^I \{A\}\, c\, \{B\}$ for every $\sigma \in \Sigma_\perp$ (or equivalently: $\mathfrak{C}[\![c]\!]A^I \subseteq B^I$).
- $\{A\}\, c\, \{B\}$ is called valid (notation: $\models \{A\}\, c\, \{B\}$) if $\models^I \{A\}\, c\, \{B\}$ for every $I \in Int$.

Hoare Logic

Goal: syntactic derivation of valid partial correctness properties

Definition (Hoare Logic)

The Hoare rules are given by

$$\text{(skip)}\ \frac{}{\{A\}\,\texttt{skip}\,\{A\}} \qquad \text{(asgn)}\ \frac{}{\{A[x \mapsto a]\}\,x\,\texttt{:=}\,a\,\{A\}}$$

$$\text{(seq)}\ \frac{\{A\}\,c_1\,\{C\} \quad \{C\}\,c_2\,\{B\}}{\{A\}\,c_1\,\texttt{;}\,c_2\,\{B\}} \qquad \text{(if)}\ \frac{\{A \wedge b\}\,c_1\,\{B\} \quad \{A \wedge \neg b\}\,c_2\,\{B\}}{\{A\}\,\texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\,\{B\}}$$

$$\text{(while)}\ \frac{\{A \wedge b\}\,c\,\{A\}}{\{A\}\,\texttt{while}\ b\ \texttt{do}\ c\,\{A \wedge \neg b\}}$$

$$\text{(cons)}\ \frac{\models (A \implies A') \quad \{A'\}\,c\,\{B'\} \quad \models (B' \implies B)}{\{A\}\,c\,\{B\}}$$

A partial correctness property is provable (notation: $\vdash \{A\}\,c\,\{B\}$) if it is derivable by the Hoare rules. In case of (while), A is called a (loop) invariant.

Here $A[x \mapsto a]$ denotes the syntactic replacement of every occurrence of x by a in A.

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Soundness of Hoare Logic I

Soundness: no wrong propositions can be derived, i.e., every (syntactically) provable partial correctness property is also (semantically) valid

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For the corresponding proof we use:

Lemma 11.1 (Substitution lemma)

For every $A \in Assn$, $x \in Var$, $a \in AExp$, $\sigma \in \Sigma$, and $I \in Int$:

$$\sigma \models^I A[x \mapsto a] \iff \sigma[x \mapsto \mathfrak{A}[\![a]\!]\sigma] \models^I A.$$

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Proof.

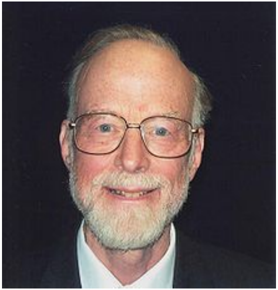
by induction over $A \in Assn$ (omitted) □

Soundness of Hoare Logic II

Theorem 11.2 (Soundness of Hoare Logic)

For every partial correctness property $\{A\}\, c\, \{B\}$,

$$\vdash \{A\}\, c\, \{B\} \implies \models \{A\}\, c\, \{B\}.$$



Tony Hoare (* 1934)

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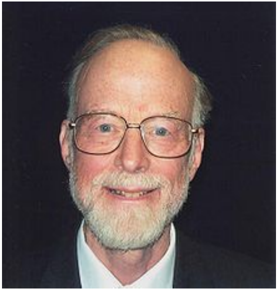

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Proof.

Let $\vdash \{A\}\, c\, \{B\}$. By induction over the structure of the corresponding proof tree we show that, for every $\sigma \in \Sigma$ and $I \in Int$ such that $\sigma \models^I A$, $\mathfrak{C}[\![c]\!]\sigma \models^I B$ (on the board).
(If $\sigma = \bot$, then $\mathfrak{C}[\![c]\!]\sigma = \bot \models^I B$ holds trivially.) □

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Soundness: only valid partial correctness properties are provable ✓

Completeness: all valid partial correctness properties are systematically derivable $\lightning$

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Theorem 11.3 (Gödel's Incompleteness Theorem)

The set of all valid assertions

$$\{A \in Assn \mid\, \models A\}$$

is not recursively enumerable, i.e., there exists no proof system for $Assn$ in which all valid assertions are systematically derivable.

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(1906–1978)

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Proof.

see [Winskel 1996, p. 110 ff] □

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Given $A \in Assn$, $\models A$ is obviously equivalent to $\{\texttt{true}\}\,\texttt{skip}\,\{A\}$. Thus the enumerability of all valid partial correctness properties would imply the enumerability of all valid assertions. □

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Remark: alternative proof (using computability theory):
$\{\mathsf{true}\}\,c\,\{\mathsf{false}\}$ is valid iff c does not terminate on any input state. But the set of all non-terminating WHILE statements is not enumerable.

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Relative Completeness of Hoare Logic I

- We will see: actual reason of incompleteness is rule

$$(\text{cons})\ \frac{\models (A \implies A') \quad \{A'\}\, c\, \{B'\} \quad \models (B' \implies B)}{\{A\}\, c\, \{B\}}$$

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$\implies$ Relative completeness

Relative Completeness of Hoare Logic II

Theorem 11.5 (Cook's Completeness Theorem)

Hoare Logic is relatively complete, i.e., for every partial correctness property $\{A\}\,c\,\{B\}$*:*

$$\models \{A\}\,c\,\{B\} \quad \Longrightarrow \quad \vdash \{A\}\,c\,\{B\}.$$

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The proof uses the following concept: assume that, e.g., $\{A\}\, c_1 ; c_2\, \{B\}$ has to be derived. This requires an intermediate assertion $C \in Assn$ such that $\{A\}\, c_1\, \{C\}$ and $\{C\}\, c_2\, \{B\}$. How to find it?

Weakest Preconditions I

Definition 11.6 (Weakest precondition)

Given $c \in Cmd$, $B \in Assn$ and $I \in Int$, the weakest precondition of B with respect to c under I is defined by:

$$wp^I[\![c, B]\!] := \{\sigma \in \Sigma_\perp \mid \mathfrak{C}[\![c]\!]\sigma \models^I B\}.$$

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Corollary 11.7

For every $c \in Cmd$, $A, B \in Assn$, and $I \in Int$:

1. $\models^I \{A\}\, c\, \{B\} \iff A^I \subseteq wp^I[\![c, B]\!]$
2. *If $A_0 \in Assn$ such that $A_0^I = wp^I[\![c, B]\!]$ for every $I \in Int$, then*
$$\models \{A\}\, c\, \{B\} \quad \iff \quad \models (A \implies A_0)$$

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$$\models \{A\}\, c\, \{B\} \quad \iff \quad \models (A \implies A_0)$$

Remark: (2) justifies the notion of weakest precondition: it is implied by every precondition A which makes $\{A\}\, c\, \{B\}$ valid

Weakest Preconditions II

Definition 11.8 (Expressivity of assertion languages)

An assertion language $Assn$ is called expressive if, for every $c \in Cmd$ and $B \in Assn$, there exists $A_{c,B} \in Assn$ such that

$$A^I_{c,B} = wp^I [\![c, B]\!]$$

for every $I \in Int$.

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$Assn$ is expressive.

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Theorem 11.9 (Expressivity of *Assn*)

Assn is expressive.

Proof.

(idea; see [Winskel 1996, p. 103 ff for details])

Given $c \in Cmd$ and $B \in Assn$, construct $A_{c,B} \in Assn$ with $\sigma \models^I A_{c,B} \iff \mathfrak{C}[\![c]\!]\sigma \models^I B$ (for every $\sigma \in \Sigma_\perp$, $I \in Int$). For example:

$$A_{\mathtt{skip},B} := B \qquad A_{x:=a,B} := B[x \mapsto a]$$
$$A_{c_1;c_2,B} := A_{c_1,A_{c_2,B}} \qquad \ldots$$

(for `while`: "Gödelization" of sequences of intermediate states) □

Relative Completeness of Hoare Logic II

The following lemma shows that weakest preconditions are "derivable":

Lemma 11.10

For every $c \in Cmd$ *and* $B \in Assn$*:*

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by structural induction over c (omitted) □

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Proof (Cook's Completeness Theorem 11.5).

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- (cons) rule $\Longrightarrow$ $\vdash \{A\}\, c\, \{B\}$

□