Semantics and Verification of Software

Lecture 5: Denotational Semantics of WHILE I (Fixpoint Semantics)

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Outline

Summary: Operational Semantics

- Formalized by evaluation/execution relations
- Inductively defined by derivation trees using structural operational rules
- Enables proofs about operational behavior of programs using structural induction on derivation trees
- Semantic functional characterizes complete input/output behavior of programs

Outline

Denotational Semantics of WHILE

- Primary aspect of a program: its "effect", i.e., input/output behavior
- In operational semantics: indirect definition of semantic functional $\mathfrak{O}[\![.]\!] : \mathit{Cmd} \to (\Sigma \dashrightarrow \Sigma)$ by execution relation
- Now: abstract from operational details
- Denotational semanics: direct definition of program effect by induction on its syntactic structure

Outline

Semantics of Arithmetic Expressions

Again: value of an expression determined by current state

Definition 5.1 (Denotational semantics of arithmetic expressions)

The (denotational) semantic functional for arithmetic expressions,

$$\mathfrak{A}[\![.]\!] : AExp \to (\Sigma \to \mathbb{Z}),$$

is given by:

$$\begin{array}{ll} \mathfrak{A}[\![z]\!]\sigma := z & \mathfrak{A}[\![a_1 + a_2]\!]\sigma := \mathfrak{A}[\![a_1]\!]\sigma + \mathfrak{A}[\![a_2]\!]\sigma \\ \mathfrak{A}[\![x]\!]\sigma := \sigma(x) & \mathfrak{A}[\![a_1 - a_2]\!]\sigma := \mathfrak{A}[\![a_1]\!]\sigma - \mathfrak{A}[\![a_2]\!]\sigma \\ & \mathfrak{A}[\![a_1 * a_2]\!]\sigma := \mathfrak{A}[\![a_1]\!]\sigma \cdot \mathfrak{A}[\![a_2]\!]\sigma \end{array}$$

Semantics of Boolean Expressions

Definition 5.2 (Denotational semantics of Boolean expressions)

The (denotational) semantic functional for Boolean expressions,

$$\mathfrak{B}[\![.]\!] : BExp \to (\Sigma \to \mathbb{B}),$$

is given by:

$$\mathfrak{B}[\![t]\!]\sigma := t$$

$$\mathfrak{B}[\![a_1{=}a_2]\!]\sigma := \begin{cases} \text{true} & \text{if } \mathfrak{A}[\![a_1]\!]\sigma = \mathfrak{A}[\![a_2]\!]\sigma \\ \text{false} & \text{otherwise} \end{cases}$$

$$\mathfrak{B}[\![a_1{>}a_2]\!]\sigma := \begin{cases} \text{true} & \text{if } \mathfrak{A}[\![a_1]\!]\sigma > \mathfrak{A}[\![a_2]\!]\sigma \\ \text{false} & \text{otherwise} \end{cases}$$

$$\mathfrak{B}[\![\neg b]\!]\sigma := \begin{cases} \text{true} & \text{if } \mathfrak{B}[\![b]\!]\sigma = \text{false} \\ \text{false} & \text{otherwise} \end{cases}$$

$$\mathfrak{B}[\![b_1 \wedge b_2]\!]\sigma := \begin{cases} \text{true} & \text{if } \mathfrak{B}[\![b_1]\!]\sigma = \mathfrak{B}[\![b_2]\!]\sigma = \text{true} \\ \text{false} & \text{otherwise} \end{cases}$$

$$\mathfrak{B}[\![b_1 \vee b_2]\!]\sigma := \begin{cases} \text{false} & \text{if } \mathfrak{B}[\![b_1]\!]\sigma = \mathfrak{B}[\![b_2]\!]\sigma = \text{false} \\ \text{true} & \text{otherwise} \end{cases}$$

Outline

Semantics of Statements I

- Now: semantic functional
$$\mathfrak{C}[\![.]\!] : \mathit{Cmd} \to (\Sigma \dashrightarrow \Sigma)$$
- Same type as operational functional
$$\mathfrak{O}[\![.]\!] : \mathit{Cmd} \to (\Sigma \dashrightarrow \Sigma)$$
(in fact, both will turn out to be the same
$\Longrightarrow$ equivalence of operational and denotational semantics)
- Inductive definition employs auxiliary functions:
  - identity on states: $\mathrm{id}_\Sigma : \Sigma \dashrightarrow \Sigma : \sigma \mapsto \sigma$
  - (strict) composition of partial state transformations:
$$\circ : (\Sigma \dashrightarrow \Sigma) \times (\Sigma \dashrightarrow \Sigma) \to (\Sigma \dashrightarrow \Sigma)$$
where, for every $f, g : \Sigma \dashrightarrow \Sigma$ and $\sigma \in \Sigma$,
$$(g \circ f)(\sigma) := \begin{cases} g(f(\sigma)) & \text{if } f(\sigma) \text{ defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$
  - semantic conditional:
$$\mathrm{cond} : (\Sigma \to \mathbb{B}) \times (\Sigma \dashrightarrow \Sigma) \times (\Sigma \dashrightarrow \Sigma) \to (\Sigma \dashrightarrow \Sigma)$$
where, for every $p : \Sigma \to \mathbb{B}$, $f, g : \Sigma \dashrightarrow \Sigma$, and $\sigma \in \Sigma$,
$$\mathrm{cond}(p, f, g)(\sigma) := \begin{cases} f(\sigma) & \text{if } p(\sigma) = \text{true} \\ g(\sigma) & \text{otherwise} \end{cases}$$

Definition 5.3 (Denotational semantics of statements)

The (denotational) semantic functional for statements,

$$\mathfrak{C}[\![.]\!] : \mathit{Cmd} \to (\Sigma \dashrightarrow \Sigma),$$

is given by:

$$\begin{aligned}
\mathfrak{C}[\![\texttt{skip}]\!] &:= \mathrm{id}_\Sigma \\
\mathfrak{C}[\![x \texttt{ := } a]\!]\sigma &:= \sigma[x \mapsto \mathfrak{A}[\![a]\!]\sigma] \\
\mathfrak{C}[\![c_1 \texttt{;} c_2]\!] &:= \mathfrak{C}[\![c_2]\!] \circ \mathfrak{C}[\![c_1]\!] \\
\mathfrak{C}[\![\texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2]\!] &:= \mathrm{cond}(\mathfrak{B}[\![b]\!], \mathfrak{C}[\![c_1]\!], \mathfrak{C}[\![c_2]\!]) \\
\mathfrak{C}[\![\texttt{while } b \texttt{ do } c]\!] &:= \mathrm{fix}(\Phi)
\end{aligned}$$

where $\Phi : (\Sigma \dashrightarrow \Sigma) \to (\Sigma \dashrightarrow \Sigma) : f \mapsto \mathrm{cond}(\mathfrak{B}[\![b]\!], f \circ \mathfrak{C}[\![c]\!], \mathrm{id}_\Sigma)$

Semantics of Statements III

Remarks:

- Definition of $\mathfrak{C}[\![c]\!]$ given by induction on syntactic structure of $c \in \mathit{Cmd}$
  - in particular, $\mathfrak{C}[\![\texttt{while}\ b\ \texttt{do}\ c]\!]$ only refers to $\mathfrak{B}[\![b]\!]$ and $\mathfrak{C}[\![c]\!]$ (and not to $\mathfrak{C}[\![\texttt{while}\ b\ \texttt{do}\ c]\!]$ again)
  - note difference to $\mathfrak{O}[\![c]\!]$:

$$\text{(wh-t)}\ \frac{\langle b, \sigma\rangle \to \text{true}\ \ \langle c, \sigma\rangle \to \sigma'\ \ \langle \texttt{while}\ b\ \texttt{do}\ c, \sigma'\rangle \to \sigma''}{\langle \texttt{while}\ b\ \texttt{do}\ c, \sigma\rangle \to \sigma''}$$

- In $\mathfrak{C}[\![c_1\,;c_2]\!] := \mathfrak{C}[\![c_2]\!] \circ \mathfrak{C}[\![c_1]\!]$, function composition $\circ$ has to be strict since non-termination of c_1 implies non-termination of $c_1\,;c_2$
- In $\mathfrak{C}[\![\texttt{while}\ b\ \texttt{do}\ c]\!] := \mathrm{fix}(\Phi)$, fix denotes a fixpoint operator (which remains to be defined)
  $\Longrightarrow$ "fixpoint semantics"

But: why fixpoints?

Why Fixpoints?

- Goal: preserve validity of equivalence

$$\mathfrak{C}[\![\texttt{while } b \texttt{ do } c]\!] \overset{(*)}{=} \mathfrak{C}[\![\texttt{if } b \texttt{ then } (c\texttt{;while } b \texttt{ do } c) \texttt{ else skip}]\!]$$

(cf. Lemma 4.3)

- Using the known parts of Def. 5.3, we obtain:

$$\begin{aligned}
&\mathfrak{C}[\![\texttt{while } b \texttt{ do } c]\!] \\
&\overset{(*)}{=} \quad \mathfrak{C}[\![\texttt{if } b \texttt{ then } (c\texttt{;while } b \texttt{ do } c) \texttt{ else skip}]\!] \\
&\overset{\text{Def. 5.3}}{=} \quad \mathrm{cond}(\mathfrak{B}[\![b]\!], \mathfrak{C}[\![c\texttt{;while } b \texttt{ do } c]\!], \mathfrak{C}[\![\texttt{skip}]\!]) \\
&\overset{\text{Def. 5.3}}{=} \quad \mathrm{cond}(\mathfrak{B}[\![b]\!], \mathfrak{C}[\![\texttt{while } b \texttt{ do } c]\!] \circ \mathfrak{C}[\![c]\!], \mathrm{id}_\Sigma)
\end{aligned}$$

- Abbreviating $f := \mathfrak{C}[\![\texttt{while } b \texttt{ do } c]\!]$ this yields:

$$f = \mathrm{cond}(\mathfrak{B}[\![b]\!], f \circ \mathfrak{C}[\![c]\!], \mathrm{id}_\Sigma)$$

- Hence f must be a solution of this recursive equation
- In other words: f must be a fixpoint of the mapping

$$\Phi : (\Sigma \dashrightarrow \Sigma) \to (\Sigma \dashrightarrow \Sigma) : f \mapsto \mathrm{cond}(\mathfrak{B}[\![b]\!], f \circ \mathfrak{C}[\![c]\!], \mathrm{id}_\Sigma)$$

(since the equation can be stated as $f = \Phi(f)$)

Well-Definedness of Fixpoint Semantics

But: fixpoint property not sufficient to obtain a well-defined semantics

Potential problems:

Existence: there does not need to exist any fixpoint. Examples:

1. $\phi_1 : \mathbb{N} \to \mathbb{N} : n \mapsto n + 1$ has no fixpoint
2. $\Phi_1 : (\Sigma \dashrightarrow \Sigma) \to (\Sigma \dashrightarrow \Sigma) : f \mapsto \begin{cases} g_1 & \text{if } f = g_2 \\ g_2 & \text{otherwise} \end{cases}$

   (where $g_1 \neq g_2$) has no fixpoint

Solution: in our setting, fixpoints always exist

Uniqueness: there might exist several fixpoints. Examples:

1. $\phi_2 : \mathbb{N} \to \mathbb{N} : n \mapsto n^3$ has fixpoints $\{0, 1\}$
2. every state transformation f is a fixpoint of $\Phi_2 : (\Sigma \dashrightarrow \Sigma) \to (\Sigma \dashrightarrow \Sigma) : f \mapsto f$

Solution: uniqueness guaranteed by choosing a special fixpoint

Outline

Characterization of $\mathrm{fix}(\Phi)$

- Let $b \in \mathit{BExp}$ and $c \in \mathit{Cmd}$
- Let $\Phi(f) := \mathrm{cond}(\mathfrak{B}[\![b]\!], f \circ \mathfrak{C}[\![c]\!], \mathrm{id}_\Sigma)$
- Let $f_0 : \Sigma \dashrightarrow \Sigma$ be a fixpoint of Φ, i.e., $\Phi(f_0) = f_0$
- Given some initial state $\sigma_0 \in \Sigma$, we will distinguish the following cases:
  1. loop `while` b `do` c terminates after n iterations ($n \in \mathbb{N}$)
  2. body c diverges in the nth iteration
     (since it contains a non-terminating `while` statement)
  3. loop `while` b `do` c itself diverges

Case 1: Termination of Loop

- Loop `while` b `do` c terminates after n iterations ($n \in \mathbb{N}$)
- Formally: there exist $\sigma_1, \ldots, \sigma_n \in \Sigma$ such that

$$\mathfrak{B}[\![b]\!]\sigma_i = \begin{cases} \text{true} & \text{if } 0 \leq i < n \\ \text{false} & \text{if } i = n \end{cases} \quad \text{and}$$

$$\mathfrak{C}[\![c]\!]\sigma_i = \sigma_{i+1} \quad \text{for every } 0 \leq i < n$$

- Now the definition $\Phi(f) := \text{cond}(\mathfrak{B}[\![b]\!], f \circ \mathfrak{C}[\![c]\!], \text{id}_\Sigma)$ implies, for every $0 \leq i < n$,

$$\begin{aligned} \Phi(f_0)(\sigma_i) &= (f_0 \circ \mathfrak{C}[\![c]\!])(\sigma_i) && \text{since } \mathfrak{B}[\![b]\!]\sigma_i = \text{true} \\ &= f_0(\sigma_{i+1}) && \text{and} \\ \Phi(f_0)(\sigma_n) &= \sigma_n && \text{since } \mathfrak{B}[\![b]\!]\sigma_n = \text{false} \end{aligned}$$

- Since $\Phi(f_0) = f_0$ it follows that

$$f_0(\sigma_i) = \begin{cases} f_0(\sigma_{i+1}) & \text{if } 0 \leq i < n \\ \sigma_n & \text{if } i = n \end{cases}$$

and hence

$$f_0(\sigma_0) = f_0(\sigma_1) = \ldots f_0(\sigma_n) = \sigma_n$$

$\Longrightarrow$ All fixpoints f_0 coincide on σ_0 (with result σ_n)!

Case 2: Divergence of Body

- Body c diverges in the nth iteration
  (since it contains a non-terminating `while` statement)
- Formally: there exist $\sigma_1, \ldots, \sigma_{n-1} \in \Sigma$ such that

$$\mathfrak{B}[\![b]\!]\sigma_i = \text{true} \qquad \text{for every } 0 \leq i < n \text{ and}$$
$$\mathfrak{C}[\![c]\!]\sigma_i = \begin{cases} \sigma_{i+1} & \text{if } 0 \leq i \leq n-2 \\ \text{undefined} & \text{if } i = n-1 \end{cases}$$

- Just as in the previous case (setting $\sigma_n := \text{undefined}$) it follows that

$$f_0(\sigma_0) = \text{undefined}$$

$\Longrightarrow$ Again all fixpoints f_0 coincide on σ_0 (with undefined result)!

Case 3: Divergence of Loop

- Loop `while` b `do` c diverges
- Formally: there exist $\sigma_1, \sigma_2, \ldots \in \Sigma$ such that

$$\begin{aligned} \mathfrak{B}[\![b]\!]\sigma_i &= \text{true} \quad \text{and} \\ \mathfrak{C}[\![c]\!]\sigma_i &= \sigma_{i+1} \quad \text{for every } i \in \mathbb{N} \end{aligned}$$

- Here only derivable:

$$f_0(\sigma_0) = f_0(\sigma_i) \quad \text{for every } i \in \mathbb{N}$$

$\Longrightarrow$ Value of $f_0(\sigma_0)$ not determined!

Summary

For $\Phi(f_0) = f_0$ and initial state $\sigma_0 \in \Sigma$, case distinction yields:

1. Loop `while` b `do` c terminates after n iterations ($n \in \mathbb{N}$)
   $\Longrightarrow$ $f_0(\sigma_0) = \sigma_n$
2. Body c diverges in the nth iteration
   $\Longrightarrow$ $f_0(\sigma_0) =$ undefined
3. Loop `while` b `do` c diverges
   $\Longrightarrow$ no condition on f_0 (only $f_0(\sigma_0) = f_0(\sigma_i)$ for every $i \in \mathbb{N}$)

- Not surprising since, e.g., for the loop `while true do skip` every $f : \Sigma \dashrightarrow \Sigma$ is a fixpoint:
  $$\Phi(f) = \text{cond}(\mathfrak{B}[\![\text{true}]\!], f \circ \mathfrak{C}[\![\texttt{skip}]\!], \text{id}_\Sigma) = f$$
- On the other hand, our operational understanding requires, for every $\sigma_0 \in \Sigma$,
  $$\mathfrak{C}[\![\texttt{while true do skip}]\!]\sigma_0 = \text{undefined}$$

Conclusion

$\text{fix}(\Phi)$ is the least defined fixpoint of Φ.