apl. Prof. Dr. Thomas Noll

Kevin van der Pol, Hao Wu

Please hand in your solutions in groups of two or three students.
You can hand them in at the start of the exercise class, Monday April 29, 10:00 in AH 2.

Hints:

Single-step semantics for Boolean expressions: $\to_1^b \subseteq (BExp \times \Sigma) \times ((BExp \times \Sigma) \cup \mathbb{B})$

$$(Bt)\ \frac{}{\langle t, \sigma\rangle \to_1^b t}$$

$$(B{=}\text{true})\ \frac{}{\langle z = z, \sigma\rangle \to_1^b \langle \text{true}, \sigma\rangle}$$

$$(B\neg 1)\ \frac{\langle b, \sigma\rangle \to_1^b \langle b', \sigma\rangle}{\langle \neg b, \sigma\rangle \to_1^b \langle \neg b', \sigma\rangle}$$

$$(B{=}\text{false})\ \frac{}{\langle z_1 = z_2, \sigma\rangle \to_1^b \langle \text{false}, \sigma\rangle}\ (z_1 \neq z_2)$$

$$(B\neg\text{false})\ \frac{}{\langle \neg\text{true}, \sigma\rangle \to_1^b \langle \text{false}, \sigma\rangle}$$

$$(B{=}1)\ \frac{\langle a_1, \sigma\rangle \to_1^a \langle a_1', \sigma\rangle}{\langle a_1 = a_2, \sigma\rangle \to_1^a \langle a_1' = a_2, \sigma\rangle}$$

$$(B\neg\text{true})\ \frac{}{\langle \neg\text{false}, \sigma\rangle \to_1^b \langle \text{true}, \sigma\rangle}$$

$$(B{=}2)\ \frac{\langle a_2, \sigma\rangle \to_1^a \langle a_2', \sigma\rangle}{\langle z_1 = a_2, \sigma\rangle \to_1^a \langle z_1 = a_2', \sigma\rangle}$$

where $b \in BExp$; $z, z_1, z_2 \in \mathbb{Z}$; $a_1, a_2 \in AExp$. (For $\geq, \vee, \wedge$ analogously.)

Exercise 1 (Single-Step Semantics): (3 Points)

In the lecture we have defined a so-called *big-step semantics* for expressions, i.e., a relation $\to \subseteq (AExp \cup BExp \cup Cmd) \times \Sigma \times (\mathbb{Z} \cup \mathbb{B} \cup \Sigma)$ which yields the value of an expression within one step: $\langle (3+5) * (5-2), \sigma\rangle \to 24$. (Thus the intermediate results of the computation are "hidden" in the derivation tree.)
Alternatively it is possible to explicitly represent the intermediate steps by defining a *single-step semantics* $\to_1 = \to_1^a \cup \to_1^b \cup \to_1^c$, such that the following expression can be evaluated as:

$$\langle (3+5) * (5-2), \sigma\rangle \to_1 \langle 8 * (5-2), \sigma\rangle \to_1 \langle 8 * 3, \sigma\rangle \to_1 \langle 24, \sigma\rangle \to_1 24.$$

Give a complete specification of the single-step relation (see *Hints* above for an example of $\to_1^b$ for Boolean expressions)

1. $\to_1^a \subseteq (AExp \times \Sigma) \times ((AExp \times \Sigma) \cup \mathbb{Z})$ for arithmetic expressions and
2. $\to_1^c \subseteq (Cmd \times \Sigma) \times (Cmd \times \Sigma)$ for commands.

Exercise 2 (Agreement of Big-Step and Single-Step Semantics): (3 Points)

Show that the big-step relation and the single-step relation on arithmetic expressions, as defined in Exercise 1, are equivalent, i.e., that for every $a \in AExp$, $\delta \in \Sigma$, and $z \in \mathbb{Z}$:

$$\langle a, \delta\rangle \to z \quad \text{iff} \quad \langle a, \delta\rangle \to_1^* z.$$

Hint: use the existential quantifier $\exists$ to assert that there exist z_1 and z_2 such that $z_1 + z_2 = z$.

Exercise 3 (repeat...until): (1+2 Points)

The **repeat**... **until** construct is similar to the **while**... **do** construct, except that the body of the loop is evaluated *at least once*. The loop ends when the guard evaluates to true. For example, with $\sigma(x) = 0$:

$$\begin{aligned} & \langle \textbf{repeat } x := x + 1 \textbf{ until } x = 2, \sigma \rangle \\ \to\ & \langle \textbf{repeat } x := x + 1 \textbf{ until } x = 2, \sigma[x \mapsto 1] \rangle \\ \to\ & \sigma[x \mapsto 2] \end{aligned}$$

(a) Define the execution relation for the **repeat**... **until** construct without using the **while**... **do** construct.
(b) Show that **repeat** c **until** $b \sim c;$ **while** $\neg b$ **do** c for any $b \in BExp, c \in Cmd$.

Exercise 4 (Infinite Loop): (2+1 Points)

(a) Show that the following proposition holds: for all states σ, σ', $\langle$**while true do skip**, $\sigma\rangle \nrightarrow \sigma'$.
(b) Show that in the single-step semantics you defined in Exercise 1, the execution of $\langle$**while true do skip**, $\sigma\rangle$ does not terminate.