3(e)
To show: $\text{fix}(\Phi)(n) = n!$ for $n \in \mathbb{Z}$.

Proof: we show $\text{fix}(\Phi)(n) \overset{(1)}{=} f_n(n) \overset{(2)}{=} n!$ for all $n \in \mathbb{Z}$, where $f_n = f_\emptyset$ for $n < 0$ and $f_n = \Phi(f_{n-1})$ for $n \geq 0$. By Knaster-Tarski, the least fixpoint of Φ is the supremum of the chain $f_\emptyset, f_0, f_1, \ldots$.

- (1) Case $n < 0$
  $n!$ is not defined for negative n.

  It remains to show that $\text{fix}(\Phi)(n)$ is not defined for negative n either. We show that $f_k(n)$ is not defined for any $k \in \{\emptyset\} \cup \mathbb{N}$. Because $\text{fix}(\Phi)$ is the supremum (the *least* upper bound) of all these functions f_k, $\text{fix}(\Phi)(n)$ is not defined either.

  Claim: $f_k(n)$ is not defined for any $k \in \{\emptyset\} \cup \mathbb{N}$. The case $f_\emptyset$ is trivial. Proof by mathematical induction for $k \in \mathbb{N}$.

  - Base: $0 \in \mathbb{N}$.
    To show: $f_0(n)$ is not defined for any $n < 0$
    Proof: let $n < 0$.

    $$\begin{aligned} f_0(n) &= \Phi(f_\emptyset)(n) \\ &= \begin{cases} 1 & \text{if } n = 1 \\ n * f_\emptyset(n-1) & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } n = 1 \\ \text{undefined} & \text{otherwise} \end{cases} \\ &= \text{undefined} \end{aligned}$$

  - Induction step: show claim for k, assuming claim holds for $k' < k$.
    Induction Hypothesis: $f_{k'}(n')$ is not defined for any $k' < k$, $n' < 0$.
    To show: $f_k(n)$ is not defined for any $n < 0$. Proof: let $n < 0$.

    $$\begin{aligned} f_k(n) &= \Phi(f_{k-1})(n) \\ &= \begin{cases} 1 & \text{if } n = 1 \\ n * f_{k-1}(n-1) & \text{otherwise} \end{cases} \\ &= \{\text{case } n < 0\} \\ &\quad n * f_{k-1}(n-1) \end{aligned}$$

    As $k - 1 < k$ and $n < 0 \Rightarrow n - 1 < 0$, the Induction Hypothesis applies, with $k' = k - 1$ and $n' = n - 1$.

    $$\begin{aligned} &= n * \text{undefined} \\ &= \text{undefined} \end{aligned}$$

- (2) Case $n < 0$

$$\begin{aligned} f_n(n) &= f_\emptyset(n) \\ &= \text{undefined}, \end{aligned}$$

and

$$n! = \text{undefined}.$$

- (1) Case $n \geq 0$

By definition of $\text{fix}(\Phi)$ as the supremum of the chain $f_\emptyset, f_0, f_1, \ldots$, it holds that $f_n \sqsubseteq \text{fix}(\Phi)$, i.e. $f_n(x) = x' \Rightarrow \text{fix}(\Phi)(x) = x'$. By (2), Case $n \geq 0$, it holds that $f_n(n) = n!$, so we conclude $\text{fix}(\Phi)(n) = n!$.

- (2) Case $n \geq 0$

By mathematical induction on n. Claim: $f_n(n) = n!$, for all $n \in \mathbb{N}$.

  - Base: $0 \in \mathbb{N}$
    To show: $f_0(0) = 0!$
    Proof: $f_0(0) = 1 = 0!$
  - Induction Step: $n \in \mathbb{N} \Rightarrow n + 1 \in \mathbb{N}$
    Induction Hypothesis: $f_n(n) = n!$
    To show: $f_{n+1}(n + 1) = (n + 1)!$
    Proof:

$$\begin{aligned} f_{n+1}(n+1) &= \Phi(f_n)(n+1) \\ &= \begin{cases} 1 & \text{if } n+1 = 0 \\ (n+1) * f_n((n+1)-1) & \text{otherwise} \end{cases} \\ &= \begin{cases} 1! & \text{if } n+1 = 0 \\ (n+1) * f_n(n) & \text{otherwise} \end{cases} \\ &= \{\text{IH}\} \\ & \begin{cases} (n+1)! & \text{if } n+1 = 0 \\ (n+1) * n! & \text{otherwise} \end{cases} \\ &= \begin{cases} (n+1)! & \text{if } n+1 = 0 \\ (n+1)! & \text{otherwise} \end{cases} \\ &= (n+1)! \end{aligned}$$