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Exercise 1 (Strict Composition): (1+1+1 Points)

Assume three different functions f, g and h are given, where

$$f, g, h : \Sigma \dashrightarrow \Sigma$$
$$\circ : (\Sigma \dashrightarrow \Sigma) \times (\Sigma \dashrightarrow \Sigma) \rightarrow (\Sigma \dashrightarrow \Sigma)$$

and $\forall \sigma_x \in \Sigma$ and $\sigma_x \neq \sigma'_x$:

$$f(\sigma_x) := \begin{cases} \sigma'_x & \text{if } \sigma_x(x) = 1 \vee 0 \\ \text{undef.} & \text{otherwise} \end{cases}$$
$$g(\sigma_x) := \begin{cases} \sigma'_x & \text{if } \sigma_x(x) = 1 \\ \sigma_x & \text{otherwise} \end{cases}$$
$$h(\sigma_x) := \begin{cases} \sigma_1 & \text{if } g(\sigma_x) = \sigma_x \\ \sigma_0 & \text{otherwise} \end{cases}$$

Compute the following composition of functions:

a) $(f \circ g \circ h)(\sigma_x)$ **b)** $(g \circ f \circ h)(\sigma_x)$ **c)** $(g \circ h \circ f)(\sigma_x)$

Exercise 2 (Denotational evaluation): (2 Points)

(a) Define the denotational semantic functional for the following arithmetic operators:

- $\mathsf{sgn}(\cdot)$, where $\mathsf{sgn}(a) = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \\ \text{undefined} & \text{otherwise} \end{cases}$
- Division: $\div$
- Absolute value: $|\cdot|$

(Note: this is not the usual definition of sgn, where $\mathsf{sgn}(0) = 1$)
(b) Use the denotational semantic functional to show that $\mathsf{sgn}(x+3)$ equals $\frac{|3+x|}{x+3}$.

Exercise 3 (Denotational Semantics of Recursive Program): (6 Points)

Consider the following recursive program for $n \in \mathbb{Z}$:

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fac(n) = if n = 0 then 1 else fac(n−1) * n ;
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a) Determine the functional $\Phi : (\mathbb{Z} \dashrightarrow \mathbb{Z}) \rightarrow (\mathbb{Z} \dashrightarrow \mathbb{Z})$ for $\mathrm{fac}(n)$, as in the lecture.

b) Show that Φ is monotonic and continuous.

c) Show that the partial order $(\mathbb{Z} \dashrightarrow \mathbb{Z}, \sqsubseteq)$ is chain complete.

d) Let $\mathfrak{D}[\![\mathrm{fac}(n)]\!]$ be defined by $\mathrm{fix}(\Phi)$. Compute the denotational semantics of fac(3).

e) Prove that the program fac calculates the factorial, i.e. $\mathrm{fix}(\Phi)(n) = n!$ for any $n \in \mathbb{Z}$.