

# Semantics and Verification of Software

## Lecture 12: Provably Correct Implementation I (Abstract Machine)

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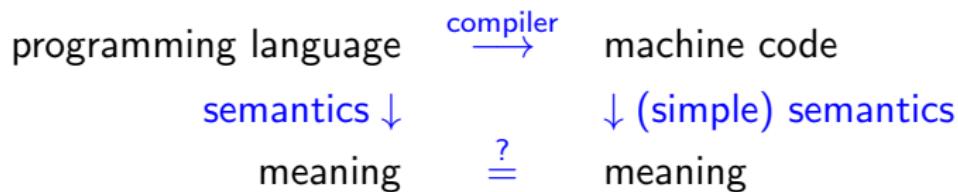
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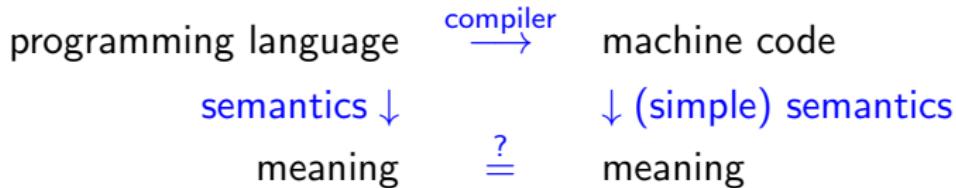
Summer Semester 2013

1 Compiler Correctness

2 The Abstract Machine

3 Properties of AM





## To do:

- ① Definition of **abstract machine**
- ② Definition (operational) **semantics of machine instructions**
- ③ Definition of **translation** WHILE  $\rightarrow$  machine code ("compiler")
- ④ **Proof:** semantics of generated machine code = semantics of original source code

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## Definition 12.1 (Abstract machine)

The abstract machine (AM) is given by

- configurations of the form  $\langle d, e, \sigma \rangle \in Cnf$  where
  - $d \in Code$  is the sequence of instructions (code) to be executed
  - $e \in Stk := (\mathbb{Z} \cup \mathbb{B})^*$  is the evaluation stack (top left)
  - $\sigma \in \Sigma := (Var \rightarrow \mathbb{Z})$  is the (storage) state

(thus  $Cnf = Code \times Stk \times \Sigma$ )

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- initial configurations of the form  $\langle d, \varepsilon, \sigma \rangle$
- final configurations of the form  $\langle \varepsilon, e, \sigma \rangle$
- code sequences  $d$  and instructions  $i$ :  
$$d ::= \varepsilon \mid i : d$$
$$i ::= PUSH(z) \mid ADD \mid MULT \mid SUB \mid$$
$$TRUE \mid FALSE \mid EQ \mid GT \mid AND \mid OR \mid NEG \mid$$
$$LOAD(x) \mid STORE(x) \mid NOOP \mid BRANCH(d, d) \mid LOOP(d, d)$$
(where  $z \in \mathbb{Z}$  and  $x \in Var$ )

## Definition 12.2 (Transition relation of AM)

The transition relation  $\triangleright \subseteq Cnf \times Cnf$  is given by

- $\langle \text{PUSH}(z) : d, e, \sigma \rangle \triangleright \langle d, z : e, \sigma \rangle$
- $\langle \text{ADD} : d, z_1 : z_2 : e, \sigma \rangle \triangleright \langle d, (z_1 + z_2) : e, \sigma \rangle$
- $\langle \text{MULT} : d, z_1 : z_2 : e, \sigma \rangle \triangleright \langle d, (z_1 * z_2) : e, \sigma \rangle$
- $\langle \text{SUB} : d, z_1 : z_2 : e, \sigma \rangle \triangleright \langle d, (z_1 - z_2) : e, \sigma \rangle$
- $\langle \text{TRUE} : d, e, \sigma \rangle \triangleright \langle d, \text{true} : e, \sigma \rangle$
- $\langle \text{FALSE} : d, e, \sigma \rangle \triangleright \langle d, \text{false} : e, \sigma \rangle$
- $\langle \text{EQ} : d, z_1 : z_2 : e, \sigma \rangle \triangleright \langle d, (z_1 = z_2) : e, \sigma \rangle$
- $\langle \text{GT} : d, z_1 : z_2 : e, \sigma \rangle \triangleright \langle d, (z_1 > z_2) : e, \sigma \rangle$
- $\langle \text{AND} : d, t_1 : t_2 : e, \sigma \rangle \triangleright \langle d, (t_1 \wedge t_2) : e, \sigma \rangle$
- $\langle \text{OR} : d, t_1 : t_2 : e, \sigma \rangle \triangleright \langle d, (t_1 \vee t_2) : e, \sigma \rangle$
- $\langle \text{NEG} : d, t : e, \sigma \rangle \triangleright \langle d, \neg t : e, \sigma \rangle$
- $\langle \text{LOAD}(x) : d, e, \sigma \rangle \triangleright \langle d, \sigma(x) : e, \sigma \rangle$
- $\langle \text{STORE}(x) : d, z : e, \sigma \rangle \triangleright \langle d, e, \sigma[x \mapsto z] \rangle$
- $\langle \text{NOOP} : d, e, \sigma \rangle \triangleright \langle d, e, \sigma \rangle$
- $\langle \text{BRANCH}(d_{\text{true}}, d_{\text{false}}) : d, t : e, \sigma \rangle \triangleright \langle d_t : d, e, \sigma \rangle$
- $\langle \text{LOOP}(d_1, d_2) : d, e, \sigma \rangle \triangleright \langle d_1 : \text{BRANCH}(d_2 : \text{LOOP}(d_1, d_2), \text{NOOP}) : d, e, \sigma \rangle$

**Remark:** more traditional machine architectures

- Variables referenced by address (and not by name)
  - configurations  $\langle d, e, m \rangle$  with memory  $m \in \mathbb{Z}^*$
  - $\text{LOAD}(x)/\text{STORE}(x)$  replaced by  $\text{GET}(n)/\text{PUT}(n)$  (where  $n \in \mathbb{N}$ )

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- `BRANCH` and `LOOP` instruction replaced by code addresses (labels) and jumping instructions
  - configurations  $\langle pc, d, e, m \rangle$  with program counter  $pc \in \mathbb{N}$
  - `BRANCH` and `LOOP` implemented by control flow, using `JUMP( $l$ )` and `JUMPFALSE( $l$ )` ( $l \in \mathbb{N}$ )

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  - `BRANCH` and `LOOP` implemented by control flow, using `JUMP(l)` and `JUMPFALSE(l)` ( $l \in \mathbb{N}$ )
- Registers for storing intermediate values  
(in place of evaluation stack  $e$ )

## Definition 12.3 (AM computations)

- A **finite computation** is a finite configuration sequence of the form  $\gamma_0, \gamma_1, \dots, \gamma_k$  where  $k \in \mathbb{N}$  and  $\gamma_{i-1} \triangleright \gamma_i$  for each  $i \in \{1, \dots, k\}$

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**Note:** a terminating computation may end in a **final configuration** ( $\langle \varepsilon, e, \sigma \rangle$ ) or in a **stuck configuration** (e.g.,  $\langle \text{ADD}, 1, \sigma \rangle$ )

## Example 12.4

For  $d := \text{PUSH}(1) : \text{LOAD}(x) : \text{ADD} : \text{STORE}(x)$  and  $\sigma(x) = 3$ , we obtain the following terminating computation:

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- ▷  $\langle \text{STORE}(x), 4, \sigma \rangle$

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- ▷  $\langle \varepsilon, \varepsilon, \sigma[x \mapsto 4] \rangle$

**Remark:** implements statement  $x := x + 1$

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The following computation loops:

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  - ▷ ...

**Remark:** implements statement `while true do skip`

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## Application: Finite computations (Def. 12.3)

Definition: a finite computation  $\gamma_0, \gamma_1, \dots, \gamma_k$  has length  $k$

Induction base: property holds for all computations of length 0

Induction hypothesis: property holds for all computations of length  $\leq k$

Induction step: property holds for all computations of length  $k + 1$

## Lemma 12.6

If  $\langle d_1, e_1, \sigma \rangle \triangleright^* \langle d', e', \sigma' \rangle$ , then

$$\langle d_1 : d_2, e_1 : e_2, \sigma \rangle \triangleright^* \langle d' : d_2, e' : e_2, \sigma' \rangle$$

for every  $d_2 \in \text{Code}$  and  $e_2 \in \text{Stk}$ .

**Interpretation:** both the code and the stack component can be extended without changing the behavior of the machine

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**Interpretation:** both the code and the stack component can be extended without changing the behavior of the machine

## Proof.

by induction on the length of the computation  
(on the board)



## Lemma 12.7

*The semantics of AM is deterministic: for all  $\gamma, \gamma', \gamma'' \in \text{Cnf}$ ,  
 $\gamma \triangleright \gamma'$  and  $\gamma \triangleright \gamma''$  imply  $\gamma' = \gamma''$ .*

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## Proof.

The successor configuration is determined by the first instruction in the code component, which is unique. □

# Another Property: Determinism

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## Proof.

The successor configuration is determined by the first instruction in the code component, which is unique.  $\square$

Thus the following function is well defined:

## Definition 12.8 (Semantics of AM)

The **semantics of an instruction sequence** is given by the mapping

$$\mathfrak{M}[\![\cdot]\!]: \text{Code} \rightarrow (\Sigma \dashrightarrow \Sigma),$$

defined by

$$\mathfrak{M}[\![d]\!](\sigma) := \begin{cases} \sigma' & \text{if } \langle d, \varepsilon, \sigma \rangle \triangleright^* \langle \varepsilon, e, \sigma' \rangle \\ \text{undefined} & \text{otherwise} \end{cases}$$