Semantics and Verification of Software

Lecture 3: Operational Semantics of WHILE II (Execution of Statements)

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Outline

Structural Induction

- Inductive set: elements are either
  - atomic or
  - obtained from atomic elements by (finite) application of certain operations
- Structural induction on $\mathbb{N}$: mathematical induction
  - induction step: $P(n) \Rightarrow P(n+1)$
- Complete induction = well-founded induction on $\mathbb{N}$
  - induction step: $P(0), P(1), \ldots, P(n) \Rightarrow P(n+1)$
  - also known as strong or course-of-values induction

Evaluation of Arithmetic Expressions

Remember: $a ::= z \mid x \mid a_1{+}a_2 \mid a_1{-}a_2 \mid a_1{*}a_2 \in AExp$

Definition (Evaluation relation for arithmetic expressions)

If $a \in AExp$ and $\sigma \in \Sigma$, then $\langle a, \sigma \rangle$ is called a configuration.

Expression a evaluates to $z \in \mathbb{Z}$ in state σ (notation: $\langle a, \sigma \rangle \to z$) if this relationship is derivable by means of the following rules:

Axioms:
$$\frac{}{\langle z, \sigma \rangle \to z} \qquad \frac{}{\langle x, \sigma \rangle \to \sigma(x)}$$

Rules:
$$\frac{\langle a_1, \sigma \rangle \to z_1 \quad \langle a_2, \sigma \rangle \to z_2}{\langle a_1{+}a_2, \sigma \rangle \to z} \quad \text{where } z := z_1 + z_2$$

$$\frac{\langle a_1, \sigma \rangle \to z_1 \quad \langle a_2, \sigma \rangle \to z_2}{\langle a_1{-}a_2, \sigma \rangle \to z} \quad \text{where } z := z_1 - z_2$$

$$\frac{\langle a_1, \sigma \rangle \to z_1 \quad \langle a_2, \sigma \rangle \to z_2}{\langle a_1{*}a_2, \sigma \rangle \to z} \quad \text{where } z := z_1 \cdot z_2$$

Evaluation of Boolean Expressions

Remember: $b ::= t \mid a_1{=}a_2 \mid a_1{>}a_2 \mid \neg b \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \in BExp$

Definition (Evaluation relation for Boolean expressions)

For $b \in BExp$, $\sigma \in \Sigma$, and $t \in \mathbb{B}$, the evaluation relation $\langle b, \sigma \rangle \to t$ is defined by the following rules:

$$\frac{}{\langle t, \sigma \rangle \to t}$$

$$\frac{\langle a_1, \sigma \rangle \to z \;\; \langle a_2, \sigma \rangle \to z}{\langle a_1{=}a_2, \sigma \rangle \to \text{true}} \qquad \frac{\langle a_1, \sigma \rangle \to z_1 \;\; \langle a_2, \sigma \rangle \to z_2}{\langle a_1{=}a_2, \sigma \rangle \to \text{false}} \text{ if } z_1 \neq z_2$$

$$\frac{\langle a_1, \sigma \rangle \to z_1 \;\; \langle a_2, \sigma \rangle \to z_2}{\langle a_1{>}a_2, \sigma \rangle \to \text{true}} \text{ if } z_1 > z_2 \qquad \frac{\langle a_1, \sigma \rangle \to z_1 \;\; \langle a_2, \sigma \rangle \to z_2}{\langle a_1{>}a_2, \sigma \rangle \to \text{false}} \text{ if } z_1 \leq z_2$$

$$\frac{\langle b, \sigma \rangle \to \text{false}}{\langle \neg b, \sigma \rangle \to \text{true}} \qquad \frac{\langle b, \sigma \rangle \to \text{true}}{\langle \neg b, \sigma \rangle \to \text{false}}$$

$$\frac{\langle b_1, \sigma \rangle \to \text{true} \;\; \langle b_2, \sigma \rangle \to \text{true}}{\langle b_1 \wedge b_2, \sigma \rangle \to \text{true}} \qquad \frac{\langle b_1, \sigma \rangle \to \text{true} \;\; \langle b_2, \sigma \rangle \to \text{false}}{\langle b_1 \wedge b_2, \sigma \rangle \to \text{false}}$$

$$\frac{\langle b_1, \sigma \rangle \to \text{false} \;\; \langle b_2, \sigma \rangle \to \text{true}}{\langle b_1 \wedge b_2, \sigma \rangle \to \text{false}} \qquad \frac{\langle b_1, \sigma \rangle \to \text{false} \;\; \langle b_2, \sigma \rangle \to \text{false}}{\langle b_1 \wedge b_2, \sigma \rangle \to \text{false}}$$

($\vee$ analogously)

Outline

Meaning of Statements

Effect of statement = modification of program state

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Example 3.1

Goal: define execution relation $\rightarrow$ such that

$$\langle \mathtt{x := 2+3}, \sigma \rangle \rightarrow \sigma[\mathtt{x} \mapsto 5]$$

where for every $\sigma \in \Sigma$, $x, y \in \mathit{Var}$, and $z \in \mathbb{Z}$:

$$\sigma[x \mapsto z](y) := \begin{cases} z & \text{if } y = x \\ \sigma(y) & \text{otherwise} \end{cases}$$

Execution of Statements

Remember:

$c ::= \texttt{skip} \mid x \texttt{ := } a \mid c_1 \texttt{;} c_2 \mid \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2 \mid \texttt{while } b \texttt{ do } c \in \mathit{Cmd}$

Execution of Statements

Remember:

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Definition 3.2 (Execution relation for statements)

For $c \in \mathit{Cmd}$ and $\sigma, \sigma' \in \Sigma$, the execution relation $\langle c, \sigma \rangle \to \sigma'$ is defined by the following rules:

$$\text{(skip)}\ \frac{}{\langle \texttt{skip}, \sigma \rangle \to \sigma} \qquad \text{(asgn)}\ \frac{\langle a, \sigma \rangle \to z}{\langle x \texttt{ := } a, \sigma \rangle \to \sigma[x \mapsto z]}$$

$$\text{(seq)}\ \frac{\langle c_1, \sigma \rangle \to \sigma' \quad \langle c_2, \sigma' \rangle \to \sigma''}{\langle c_1 \texttt{;} c_2, \sigma \rangle \to \sigma''} \qquad \text{(if-t)}\ \frac{\langle b, \sigma \rangle \to \text{true} \quad \langle c_1, \sigma \rangle \to \sigma'}{\langle \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2, \sigma \rangle \to \sigma'}$$

$$\text{(if-f)}\ \frac{\langle b, \sigma \rangle \to \text{false} \quad \langle c_2, \sigma \rangle \to \sigma'}{\langle \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2, \sigma \rangle \to \sigma'} \qquad \text{(wh-f)}\ \frac{\langle b, \sigma \rangle \to \text{false}}{\langle \texttt{while } b \texttt{ do } c, \sigma \rangle \to \sigma}$$

$$\text{(wh-t)}\ \frac{\langle b, \sigma \rangle \to \text{true} \quad \langle c, \sigma \rangle \to \sigma' \quad \langle \texttt{while } b \texttt{ do } c, \sigma' \rangle \to \sigma''}{\langle \texttt{while } b \texttt{ do } c, \sigma \rangle \to \sigma''}$$

An Execution Example

Example 3.3

- $c := \texttt{y := 1; while}\underbrace{\neg\texttt{(x=1)}}_{b}\texttt{do}\underbrace{\underbrace{\texttt{y := y*x}}_{c_1}\texttt{;}\underbrace{\texttt{x := x-1}}_{c_2}}_{c_0}$
- Claim: $\langle c, \sigma\rangle \to \sigma_{1,6}$ for every $\sigma \in \Sigma$ with $\sigma(\texttt{x}) = 3$
- Notation: $\sigma_{i,j}$ means $\sigma(\texttt{x}) = i$, $\sigma(\texttt{y}) = j$
- Derivation tree: on the board

Non-Terminating Statements

Corollary 3.4

The execution relation for statements is not total, i.e., there exist $c \in Cmd$ and $\sigma \in \Sigma$ such that $\langle c, \sigma \rangle \to \sigma'$ for no $\sigma' \in \Sigma$.

Non-Terminating Statements

Corollary 3.4

The execution relation for statements is not total, i.e., there exist $c \in Cmd$ *and* $\sigma \in \Sigma$ *such that* $\langle c, \sigma \rangle \to \sigma'$ *for no* $\sigma' \in \Sigma$.

Proof.

Counterexample: $c = \texttt{while true do skip}$
(by contradiction; on the board) □

Outline

Determinism of Execution Relation I

This operational semantics is well defined in the following sense:

Theorem 3.5

The execution relation for statements is deterministic, i.e., whenever $c \in Cmd$ *and* $\sigma, \sigma', \sigma'' \in \Sigma$ *such that* $\langle c, \sigma \rangle \to \sigma'$ *and* $\langle c, \sigma \rangle \to \sigma''$*, then* $\sigma' = \sigma''$.

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The proof is based on the corresponding result for expressions.

Determinism of Evaluation Relations

Lemma 3.6

1. *For every* $a \in AExp$, $\sigma \in \Sigma$, *and* $z, z' \in \mathbb{Z}$:
   $\langle a, \sigma \rangle \to z$ *and* $\langle a, \sigma \rangle \to z'$ *implies* $z = z'$.
2. *For every* $b \in BExp$, $\sigma \in \Sigma$, *and* $t, t' \in \mathbb{B}$:
   $\langle b, \sigma \rangle \to t$ *and* $\langle b, \sigma \rangle \to t'$ *implies* $t = t'$.

Determinism of Evaluation Relations

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   $\langle b, \sigma \rangle \to t$ *and* $\langle b, \sigma \rangle \to t'$ *implies* $t = t'$.

Remarks:

- Lemma 3.6(1) is not implied by Lemma 2.6
  ("$\sigma|_{FV(a)} = \sigma'|_{FV(a)} \Rightarrow (\langle a, \sigma \rangle \to z \iff \langle a, \sigma' \rangle \to z)$")!

  The latter just implies

  $$\{z \in \mathbb{Z} \mid \langle a, \sigma \rangle \to z\} = \{z \in \mathbb{Z} \mid \langle a, \sigma' \rangle \to z\}$$

  while Lemma 3.6(1) states that

  $$|\{z \in \mathbb{Z} \mid \langle a, \sigma \rangle \to z\}| \leq 1.$$

- Lemma 3.6 can be shown by induction on the structure of expressions.

Excursus: Proof by Structural Induction V

Application: Boolean expressions (Def. 1.2)

Definition: $BExp$ is the least set which

- contains the truth values $t \in \mathbb{B}$ and, for every $a_1, a_2 \in AExp$, $a_1{=}a_2$ and $a_1{>}a_2$, and
- contains $\neg b_1$, $b_1 \wedge b_2$ and $b_1 \vee b_2$ whenever $b_1, b_2 \in BExp$

Induction base: $P(t)$, $P(a_1{=}a_2)$ and $P(a_1{>}a_2)$ holds
(for every $t \in \mathbb{B}$, $a_1, a_2 \in AExp$)

Induction hypothesis: $P(b_1)$ and $P(b_2)$ holds

Induction step: $P(\neg b_1)$, $P(b_1 \wedge b_2)$ and $P(b_1 \vee b_2)$ holds

Excursus: Proof by Structural Induction V

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Induction hypothesis: $P(b_1)$ and $P(b_2)$ holds

Induction step: $P(\neg b_1)$, $P(b_1 \wedge b_2)$ and $P(b_1 \vee b_2)$ holds

Proof (Lemma 3.6).

1. by structural induction on a (omitted)
2. by structural induction on b (omitted)

□

Determinism of Execution Relation II

- How to prove that $\langle c, \sigma \rangle \to \sigma'$ is deterministic (Theorem 3.5)?
- Idea: use induction on the syntactic structure of c

Excursus: Proof by Structural Induction VI

Application: syntax of WHILE statements (Def. 1.2)

Definition: Cmd is the least set which

- contains $\texttt{skip}$ and, for every $x \in Var$ and $a \in AExp$, $x \texttt{ := } a$, and
- contains $c_1 \texttt{;} c_2$, $\texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2$ and $\texttt{while } b \texttt{ do } c_1$ whenever $b \in BExp$ and $c_1, c_2 \in Cmd$

Induction base: $P(\texttt{skip})$ and $P(x \texttt{ := } a)$ holds
(for every $x \in Var$ and $a \in AExp$)

Induction hypothesis: $P(c_1)$ and $P(c_2)$ holds

Induction step: $P(c_1 \texttt{;} c_2)$, $P(\texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2)$ and $P(\texttt{while } b \texttt{ do } c_1)$ holds
(for every $b \in BExp$)

Determinism of Execution Relation III

- But: proof of Theorem 3.5 fails!

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- Problematic case:

$$c = \texttt{while}\ b\ \texttt{do}\ c_0 \quad \text{where} \quad \langle b, \sigma \rangle \to \text{true}$$

Determinism of Execution Relation III

- But: proof of Theorem 3.5 fails!
- Problematic case:

$$c = \texttt{while}\ b\ \texttt{do}\ c_0 \quad \text{where} \quad \langle b, \sigma\rangle \to \text{true}$$

- Here $\langle c, \sigma\rangle \to \sigma'$ and $\langle c, \sigma\rangle \to \sigma''$ require existence of $\sigma_1, \sigma_2 \in \Sigma$ such that

$$\text{(wh-t)}\frac{\langle b, \sigma\rangle \to \text{true}\ \ \langle c_0, \sigma\rangle \to \sigma_1\ \ \langle c, \sigma_1\rangle \to \sigma'}{\langle c, \sigma\rangle \to \sigma'}$$

and

$$\text{(wh-t)}\frac{\langle b, \sigma\rangle \to \text{true}\ \ \langle c_0, \sigma\rangle \to \sigma_2\ \ \langle c, \sigma_2\rangle \to \sigma''}{\langle c, \sigma\rangle \to \sigma''}$$

Determinism of Execution Relation III

- But: proof of Theorem 3.5 fails!
- Problematic case:

$$c = \texttt{while}\ b\ \texttt{do}\ c_0 \quad \text{where} \quad \langle b, \sigma\rangle \to \text{true}$$

- Here $\langle c, \sigma\rangle \to \sigma'$ and $\langle c, \sigma\rangle \to \sigma''$ require existence of $\sigma_1, \sigma_2 \in \Sigma$ such that

$$\text{(wh-t)}\frac{\langle b, \sigma\rangle \to \text{true}\ \ \langle c_0, \sigma\rangle \to \sigma_1\ \ \langle c, \sigma_1\rangle \to \sigma'}{\langle c, \sigma\rangle \to \sigma'}$$

and

$$\text{(wh-t)}\frac{\langle b, \sigma\rangle \to \text{true}\ \ \langle c_0, \sigma\rangle \to \sigma_2\ \ \langle c, \sigma_2\rangle \to \sigma''}{\langle c, \sigma\rangle \to \sigma''}$$

- c_0 proper substatement of c
  $\Rightarrow$ induction hypothesis yields $\sigma_1 = \sigma_2$

Determinism of Execution Relation III

- But: proof of Theorem 3.5 fails!
- Problematic case:

$$c = \texttt{while}\ b\ \texttt{do}\ c_0 \quad \text{where} \quad \langle b, \sigma\rangle \to \text{true}$$

- Here $\langle c, \sigma\rangle \to \sigma'$ and $\langle c, \sigma\rangle \to \sigma''$ require existence of $\sigma_1, \sigma_2 \in \Sigma$ such that

$$\text{(wh-t)}\frac{\langle b, \sigma\rangle \to \text{true}\ \ \langle c_0, \sigma\rangle \to \sigma_1\ \ \langle c, \sigma_1\rangle \to \sigma'}{\langle c, \sigma\rangle \to \sigma'}$$

and

$$\text{(wh-t)}\frac{\langle b, \sigma\rangle \to \text{true}\ \ \langle c_0, \sigma\rangle \to \sigma_2\ \ \langle c, \sigma_2\rangle \to \sigma''}{\langle c, \sigma\rangle \to \sigma''}$$

- c_0 proper substatement of c
  $\Rightarrow$ induction hypothesis yields $\sigma_1 = \sigma_2$
- c not proper substatement of $c \Rightarrow$ conclusion $\sigma' = \sigma''$ invalid!

Excursus: Proof by Structural Induction VII

Application: derivation trees of execution relation (Def. 3.2)

(skip): for every $\sigma \in \Sigma$, $\dfrac{}{\langle \texttt{skip}, \sigma \rangle \to \sigma}$ is a derivation tree for $\langle \texttt{skip}, \sigma \rangle \to \sigma$

(asgn): if s is a derivation tree for $\langle a, \sigma \rangle \to z$ (Def. 2.2), then $\dfrac{s}{\langle x := a, \sigma \rangle \to \sigma[x \mapsto z]}$ is a derivation tree for $\langle x := a, \sigma \rangle \to \sigma[x \mapsto z]$

(seq): if s_1 and s_2 are derivation trees for $\langle c_1, \sigma \rangle \to \sigma'$ and, respectively, $\langle c_2, \sigma' \rangle \to \sigma''$, then $\dfrac{s_1 \quad s_2}{\langle c_1 ; c_2, \sigma \rangle \to \sigma''}$ is a derivation tree for $\langle c_1 ; c_2, \sigma \rangle \to \sigma''$

(if-t): if s_1 and s_2 are derivation trees for $\langle b, \sigma \rangle \to \texttt{true}$ (Def. 2.7) and, respectively, $\langle c_1, \sigma \rangle \to \sigma'$, then $\dfrac{s_1 \quad s_2}{\langle \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2, \sigma \rangle \to \sigma'}$ is a derivation tree for $\langle \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2, \sigma \rangle \to \sigma'$

(if-f): analogously

(wh-t): if s_1, s_2 and s_3 are derivation trees for $\langle b, \sigma \rangle \to \texttt{true}$ (Def. 2.7), $\langle c, \sigma \rangle \to \sigma'$ and $\langle \texttt{while } b \texttt{ do } c, \sigma' \rangle \to \sigma''$, respectively, then $\dfrac{s_1 \quad s_2 \quad s_3}{\langle \texttt{while } b \texttt{ do } c, \sigma \rangle \to \sigma''}$ is a derivation tree for $\langle \texttt{while } b \texttt{ do } c, \sigma \rangle \to \sigma''$

(wh-f): if s is a derivation tree for $\langle b, \sigma \rangle \to \texttt{false}$ (Def. 2.7), then $\dfrac{s}{\langle \texttt{while } b \texttt{ do } c, \sigma \rangle \to \sigma}$ is a derivation tree for $\langle \texttt{while } b \texttt{ do } c, \sigma \rangle \to \sigma$

Excursus: Proof by Structural Induction VIII

Application: derivation trees of execution relation (continued)

Induction base: $P\left(\dfrac{}{\langle \texttt{skip}, \sigma\rangle \to \sigma}\right)$ holds for every $\sigma \in \Sigma$, and $P(s)$ holds for every derivation tree s for an arithmetic or Boolean expression.

Induction hypothesis: $P(s_1)$, $P(s_2)$ und $P(s_3)$ holds.

Induction step: it also holds that

(asgn): $P\left(\dfrac{s_1}{\langle x \texttt{:=} a, \sigma\rangle \to \sigma[x \mapsto z]}\right)$

(seq): $P\left(\dfrac{s_1 \quad s_2}{\langle c_1 \texttt{;} c_2, \sigma\rangle \to \sigma''}\right)$

(if-t): $P\left(\dfrac{s_1 \quad s_2}{\langle \texttt{if } b \texttt{ then } c_1 \texttt{ else } c_2, \sigma\rangle \to \sigma'}\right)$

(if-f): analogously

(wh-t): $P\left(\dfrac{s_1 \quad s_2 \quad s_3}{\langle \texttt{while } b \texttt{ do } c, \sigma\rangle \to \sigma''}\right)$

(wh-f): $P\left(\dfrac{s_1}{\langle \texttt{while } b \texttt{ do } c, \sigma\rangle \to \sigma}\right)$

Determinism of Execution Relation IV

Proof (Theorem 3.5).

To show:

$$\langle c, \sigma \rangle \to \sigma', \langle c, \sigma \rangle \to \sigma'' \Rightarrow \sigma' = \sigma''$$

(by structural induction on derivation trees; on the board) □